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# **REFLECTIONS ON THE INTRODUCTION OF ALGEBRA IN GRADE 8: A TEACHER'S PERSPECTIVE.**

by

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**Declaration:**

This work has not been previously submitted in whole, or in part, for the award of any degree. It is my own work. Each significant contribution to, and quotation in, this dissertation from the work, or works, of other people has been attributed, and has been cited and referenced.

Signature:

Signed by candidate

Date: 1 September 2003.

Signature removed

**Acknowledgments**

I would like to express my appreciation to the following people who helped me with this dissertation. Thank you -

To God who is the source of life and creativity.

To my husband, Mike whose tireless encouragement got me through to the end.

To my children, Alison, Greg, Chris and Lesley who have been patient while their mother worked.

To my supervisor, Chris Breen for his understanding of where I was and where I wanted to go.

To my Grade 8 class of 2002, for teaching me about algebra.

## ABSTRACT

This dissertation looks at the early learning of algebra from a classroom teacher's perspective. There are three aspects that inform the topic: the actual classroom, the literature on beginning algebra and a sample of some of the current textbooks used in South Africa.

From my reflections on my own teaching of a Grade 8 class, I was able to identify those definitions and beliefs about algebra which were shaping my teaching. Based on the literature I classified the approach to algebra that I was foregrounding in my classroom. I saw that this traditional methodology was limiting the scope of my students' outlook on algebra. I discovered more approaches to the early learning of algebra which could broaden my students' view of the subject. This dissertation presents the following four approaches to introducing algebra: generalisation; problem solving; modeling and functions. The traditional approach is discussed as a part of generalisation. My research shows that elements of all of these approaches need to be included when introducing algebra. Often the approach to algebra is largely determined by the choice of textbook used. My analysis shows that many widely used textbooks tend to emphasise only one of the above four approaches. The dissertation notes the mistakes that are typically made by following each approach in addition to the haphazard errors made by students who are starting to learn algebra. Further light on the mistakes made by students is given by looking at how standard algebraic symbols developed over centuries of time.

A teacher who is aware of the processes of her students learning algebra should be able to see beyond the mistakes that are made. She should encourage the learning of algebra as giving a broad conception of its disciplines and applications, and not as a narrow set of prescribed learned manipulations.

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## CHAPTER 1 INTRODUCTION

I am a middle-aged mathematics teacher, but I have not taught for all of the 26 years that I have been out of university. I majored in mathematics at the University of Cape Town. I then did the Higher Diploma of Education. Initially I taught Mathematics, General and Physical Science for four years and then I went into full-time mothering for 17 years with occasional forays into the classroom for a few weeks at a time. My husband was a mathematics teacher as well and I would mark his classes' tests to keep up with the mathematics. For three years prior to coming back into full-time teaching I did some part-time work in a distance-learning context, where my role was to guide students and correct their assignments. For more than a decade I have also tutored a disabled girl in mathematics from grade 8 to 3<sup>rd</sup> year university level. Three of my four children have finished school and each of them has had his or her own style of learning mathematics. My fourth child was in Grade 8 while I researched this topic. I had then been back in full-time teaching for three years teaching only mathematics.

When I came back into full-time teaching as a mature mother, I saw my teaching through different eyes. Each of the children I taught could have been my child. I found myself being more thoughtful about their learning and their struggles. I asked myself a lot of questions concerning my teaching as I got thrown back into the hurly-burly of school life. One of the foremost questions was: why do students struggle with algebra?

### **Why algebra?**

More specifically, I asked what is algebra and how should I teach it? Formal algebra is one of the critical topics in mathematics and in South Africa it is typically introduced fairly early in the Grade 8 year, to students aged 12 - 13 years. Some children take to it easily, but for many this is the place where they find that they are not coping with their mathematics. I had always followed the approach of the prescribed textbook when introducing algebra, but I now found myself asking whether this was indeed the best approach. I wanted to know what was the most

satisfactory way to teach algebra to my Grade 8s so that it would not be such a stumbling block to them.

### **Plan of Action**

With the above in mind, the aim of this research was to focus on the nature of algebra and the ways in which it could be introduced in a Grade 8 classroom. My purpose was to determine whether to change the content of my teaching – the selection of the exercises and approaches. For this I needed to be clear about the nature of the subject. I also wanted to consider the importance of algebra in the broader field of mathematics.

I started the research by looking at my Grade 8 mathematics class and the process of their learning of algebra during 2002. This is reported in chapter 2. My focus was on what they were taught. I taught algebra in the way I usually did, using the textbook as my guide. I stood back and looked at how I taught algebra asking what the way in which I taught it said about my understanding of the nature of algebra. At the conclusion of my research, I hoped that I would be able to look back from this base level and see whether there had been a change in my view of algebra, which might in turn lead to a change in the way, I introduced algebra in the future.

The following two chapters are on aspects of the literature on the topic. The first deals with ‘what is algebra?’ while the second examines ‘how to teach algebra.’

Chapter 3 reviews the historical development of symbolic notation as well as some definitions of algebra. I needed this background to give me a conviction of the place of algebra in the broad context of mathematics. My beliefs as a teacher about what algebra is do have an impact on how I teach algebra, showing its importance even though some students struggle with it.

Chapter 4 summarises what the literature has to say about the approaches to learning algebra. The focus is on the introductory algebra programme and four approaches were identified. The chapter includes typical mistakes that early learners might make as a result of following a particular approach, since each approach will naturally lead to typical errors.

Chapter 5 analyses a selection of six current Grade 8 mathematics textbooks. Each textbook has its own programme on how to introduce algebra. I analysed the algebra content of each book and its view of algebra. Then I related each book to the four approaches, which were identified in the literature.

Chapter 6 concludes the dissertation with a justification for the use of a varied approach when introducing algebra. This recommendation rests on a view of the nature of algebra, which sees the importance of its varied facets for a beginning learner.

University of Cape Town



## CHAPTER 2 WHAT HAPPENED – TEACHING ALGEBRA IN 2002

This chapter is an account of the way in which I introduced algebra to my Grade 8 class at the start of 2002. I have recorded the events on a weekly basis and have included what the students were taught.

### **Background to my observation.**

I teach in an all girls' school in the leafy suburbs of Cape Town. Grade 8 is the first year at the school and the girls come from a variety of junior schools. There are five Grade 8 classes in the school with about 30 students in each class. In 2002, I taught one class and two other teachers taught two classes each. So there were three of us teaching Grade 8 Mathematics in 2002. We all taught according to a year plan, which we had decided on at the end of the previous year. The classes had four 50-minute lessons per week. It takes some of the girls about six months to settle into the new environment of high school and it is during those six months that algebra is introduced.

My particular class had 30 students. They came from 16 different primary schools, ten of which were local, three were from other towns and three were in neighbouring countries. The Grade 8 classes are not streamed and so my class had a wide range of abilities as well as heterogeneous backgrounds. I spent the first weeks of the year consolidating arithmetic. I revised multiplication tables, bonds and fractions to check their skill with and understanding of number. At this stage I identified six students in my class whose numerical skills were much weaker than those of the rest of the class. This could be a problem as Skemp (1971, p.35) says: "One can understand the geography of Africa even if one has missed that of Europe; ... But to understand algebra without ever having really understood arithmetic is an impossibility, for much of the algebra we learn at school is generalised arithmetic. Since many people learn to do the manipulations of arithmetic with very imperfect understanding of the underlying principles, it is small wonder that mathematics remain(s) a closed book to them." There were another six or seven (not as clearly identifiable) whose numerical skills were excellent.

My established practice in teaching has been for students to learn the skills so that when a new topic is taught they can concentrate on the topic and not get sidetracked by having to master skillbased work. For example, I thoroughly revised fractions with numbers only, before I started to teach fractions with letters. I could then focus on the algebra and not how to manipulate fractions when I taught fractions with algebra. This approach has been criticised for teaching the parts but never integrating the whole. It is like learning to dribble, kick and head a ball without having any idea of a game of soccer (Mason, Graham, Pimm and Gowar, 1985). Senseless manipulative drill is not popular with today's mathematics policy makers, but when you put manipulation into a broader programme, which teaches algebra in the context of generalised arithmetic to model and solve problems, then with skilled teaching the bits can be made to come together.

**Plan of action and collection of data:**

I told my class that I was going to write about their learning algebra. I said that I would tell them what I was thinking and feeling as I taught and that I wanted them to tell me how they were thinking and feeling about learning. At my request, a few of them wrote their thoughts down in their books. I collected these periodically and responded in a non-directive but affirming way before returning the books. I also analysed their tests and examinations to see the types of mistakes that they made as beginning algebra learners. The account below outlines our journey over two periods totaling 11 weeks during which I tried to change my introduction without being very clear where I was heading or about what I would do about the obstacles I should have anticipated along my chosen path.

My approach in the rest of this chapter is to look first at what happened each week, second to characterise my purpose and third to identify the method used (the style of teaching and the approach). Then I comment on the classroom dynamic, sometimes including comments from the students. The periods were 50 minutes long and the controlled tests were given for a whole period to all the Grade 8 classes. Copies of the worksheets and the tests are in the appendix.

**2.1. The story of week 1: Investigating functions. (Three periods)**

I decided to let the students investigate number patterns as my introduction to algebra. I gave them a variety of patterns to work through on a worksheet. (pp. 98 – 100 in appendix 2) They worked in pairs. They had to complete the pattern and generalise it. Thus each flow diagram and table had a letter in the last place. The aim was for them to see the pattern, write it in words and then do the same thing to the 'letter'. Most of them had come across similar patterns before at junior school, but these went a few steps deeper as I asked them to complete the pattern with a letter as well as asking them to write the relationship in words.

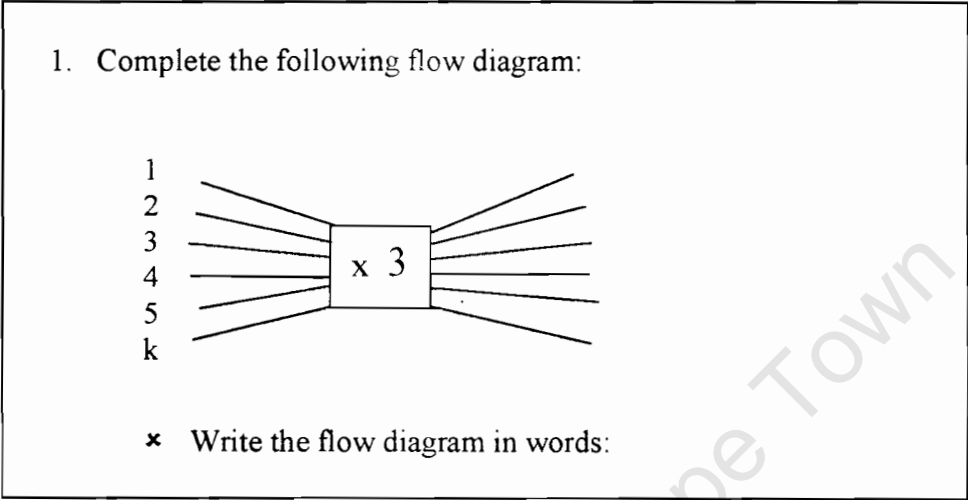


Figure 1: Exercises on early algebra

The next day we talked about algebra along the following lines. "We use letters because they are convenient and familiar symbols to put in place of numbers when the number is unknown. Algebra is a shorthand way of expressing the problem. Using letters generalises the situation. Algebra is a language with a grammar like a spoken language." We looked at operations, which are like verbs and letters, which act like nouns. Then we wrote expressions in algebra. (p. 101 in appendix 2)

The difference between three and one.	$3 - 1$
The square of 3	$3^2$
Three times a number.	$3 \times p$
4 divided by a number	$4 \div a$ or $\frac{4}{a}$

Figure 2: Verbal expressions translated into algebraic form.

Then they translated English into algebra and algebra into English.

Take a number, multiply it by 2 and add 3	$\rightarrow$	$p \times 2 + 3$	or	$2p + 3$
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Figure 3: Translation from English to algebra

The third day I told them some of the conventions of algebra like  $3p$  as opposed to  $3 \times p$  and  $\frac{4}{a}$  for  $4 \div a$ . We looked at the different ways that they wrote the formula for a rectangle that they had been taught in junior school.

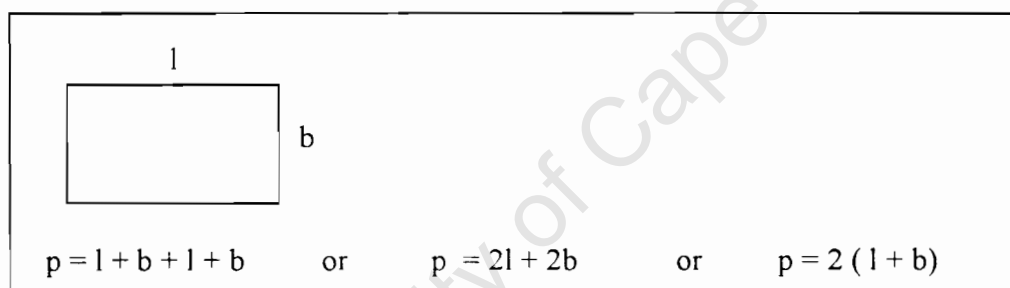


Figure 4: Formula for perimeter of a rectangle

This led on to a discussion on equivalent expressions. I used some input-output tables to help them identify equivalent expressions. (p. 102 in appendix 2)

x	1	2	3	5	10
$2x + 5x$					
$3x + 4x$					
$12x - 5$					
$7x$					
$6x + x$					
$9x - 2x$					

(a) What do you notice in the table?

(b) Determine the value of  $2x + 5x$  if  $x = 19$ . Discuss your method.

(c) Determine the value of  $x$  if  $9x - 2x = 35$ . Discuss your method.

Figure 5: Table to show equivalent forms

I let them complete this last exercise for homework so that the slower ones could catch up.

**The purpose**

At that time, early in 2002, I wrote of my plan:

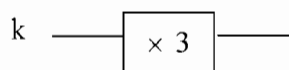
"I am going to introduce algebra with patterns. They will use symbols/letters as a form of generalising a pattern. By writing the relationship between the variables in words and then seeing the relationship in variables (i.e. expressed algebraically) I hope to make them aware of the value of algebra, namely to write mathematical expressions and equations in a concise form. I am going to try to develop the concepts from an operational conception to a structural conception. It will also introduce them to the conventions of algebra e.g.  $2 \times 3 = 6$  and  $2 \times n = 2n$ ." I used some of the ideas from Mathematics at Work for the exercises I developed for this introduction (Human et al., 2000a).

### The method

I gave the worksheets to the students and then let them work at their own pace. I walked around, observing, listening and making comments when I saw the students in trouble. Essentially I was using a functional approach, whereby looking at the output values, the students could see the relationship between the input and output values. The ability to write this relationship in letters would mean that the students were generalising as well as handling the functional relationship.

### Comments on what happened in the classroom

When I gave them the worksheet (see figure 1) I did not comment on the presence of the letter k, but when I was asked what they had to do with the letter I encouraged them to look at the verbal description of the pattern. There were different responses to the k on the spider diagram, as the students called question 1. Five answers given by students to this question



were as follows: 12, kkk, 33, x and  $3 \times k$ . In the class discussion the students gave the following explanations:

12: "I wrote 12 because in the column of 1,2,3,4,5 k comes next so it must be 6 if it follows the pattern."

kkk: "It is three k's"

33: "k is the 11<sup>th</sup> letter of the alphabet so it must stand for 11, so 3 times 11 is 33."

x: "because it is unknown."

$3 \times k$ : "all the others are timesed by three."

I was trying to see what responses I would get, whether any of them could see that algebra generalises the rule. I felt that this would get them to the next stage as the patterns all demanded a letter representation of the pattern.

Many in the class filled in all the numbers of all the questions, but left out the letters and the writing in words. When this was pointed out to them, they wanted to know whether they really *had* to write it in words. I told them that expressing it in words showed that they understood the

process. It required more effort than filling in numbers. For writing expressions using the 'verbs' in algebra, there were a few of the type that only had numbers. (see figure 2) Many of the students wrote the expressions without 'getting the answer'. They left it as ' $3 - 1$ '. Jane asked, "Do you mean that those are statements?" This indicated to me that she had an awareness that an algebraic expression could be left in an unsimplified form.

When it came to equivalent expressions, there were five girls who had difficulty in filling in the table, as their arithmetic skills were so poor. There were about eight of the class who once they saw the pattern emerging, filled in the rest of the answers without calculating them. There was a lot of non-mathematical chattering (talking about other things) going on as the students worked at different speeds. Many of the students worked through the whole worksheet leaving out the parts that they could not see an answer to immediately. They were then reluctant to go back to the questions they had left out. They would rather wait until I came around to them or until I explained it to the whole class. The students who saw patterns quickly, whizzed through the exercises and wanted to know what the issue was. Those students whose numerical skills were weak got bogged down on the arithmetic which they often got wrong, with the result that they could not see that there were patterns at all. I felt that some of the students needed more of a challenge to think through what they had written. However I did not have anything challenging at hand to give to take them further.

### **Reflection – looking back**

Reflecting on their reluctance to write the flow diagram in words, I thought that they might have felt that it was such an obvious statement, and therefore a waste of time to write. Arithmetic with small numbers is so automatic to students that they often fail to see what the question is about when we ask interpretive questions about numbers such as: "Describe in words how you get the next number in a sequence." I also struggled with classroom management with the investigative approach as the students worked at different rates with varying abilities.

### **2.2. The story of week 2: Equivalent expressions. (Two periods plus one test period)**

After the weekend, I checked the solutions with the class so that we could all draw the required conclusions from the corrected tables. To test their understanding of equivalent expressions I

asked them to generate their own equivalent expressions for  $19x$ . I then spoke to them about the conventions and rules of algebra. They made a list of the conventions in their workbooks. The conventions that I taught them for the first algebra chapter were:

1. Do not use a  $\times$  or  $\div$  sign in algebra:  $3 \times p = 3p$ ,  $\frac{4}{a}$  for  $4 \div a$
2. Always write the number in front of the letter:  $3ab$
3. But we leave out the 1 in front:  $1p = p$
4.  $k \times k = k^2$ ,  $a \times a \times a = a^3$
5.  $3a = 3 \times a$  or  $a + a + a$
6. Write terms in alphabetical order  $cab$  should be  $abc$
7. Multiplication is a shorthand of addition.  $a + a + a + a = 4a$

They had an exercise out of the textbook to practice the conventions for homework.

**Exercise 2**  
Write the following in their simplest form:

1. $3 \times a$	2. $a \times b \times c$
3. $a \times 2$	4. $3 \times a \times 4 \times b$

**Exercise 5**  
Write the following in their simplest form:

1. $a + a + a$	2. $5a + 7a$
3. $a + 3a + a$	4. $2a - a$

Figure 6: practising the conventions

They learned how to multiply using powers. The first of the controlled tests was written in this week. (pp. 103 - 104 in appendix 2)

### The purpose

The equivalent expressions should help them to see why like terms can be added. The work given from the textbook was to let them learn the manipulation skills of algebra.



### **The method**

The textbook that I was using was Just Mathematics 6 (Fitton , de Jager & Blake, 1991). It had exercises for the students to practice the skill. As the student made a mistake, I discussed with her why the mistake was made and then I explained the situation again or corrected any misconception. If I were asked the same question a few times by other students, I would then explain the problem again to the whole class. As I felt I had a good relationship with most of the girls in my class, this worked reasonably well. The biggest problem was the time factor. I often did not get around to all of those who had problems and this in a class of only 30 girls.

### **Comments on what happened in the classroom**

I asked them to write equivalent expressions to  $19x$  on the chalkboard. Many came up and wrote statements like

$$19x = 10x + 9x$$

Leila observed that everyone was writing two terms. She wanted to know whether that was that the only way to write it. At my invitation she then came up and wrote

$$19x = 5x + 4x + 10x.$$

Inspired by that variation someone else varied the operations used and wrote

$$19x = 2x \times 9x + 1x.$$

I briefly showed them that when we multiply  $2x$  by  $9x$  in algebra we get  $18x^2$  and not  $18x$ . This was interesting in that it showed a conceptual grasp of equivalence but only a partial skill in manipulation.

After working with the tables to find equivalent expressions, I decided to change my teaching style. I would teach using the chalkboard and follow the textbook's approach and not use the discovery method. I made a deliberate decision to teach using the textbook and then supplement this with the other material when needed. I felt that the varied abilities and backgrounds of the students in the class called for me to develop a common framework which would enable even the weaker students to have this secure reference point to revert to when they got stuck. I hoped that these conventions would give the students a sense of security in that they had something on paper that they could refer to. It was something they could learn. They now had an unambiguous set of rules to refer to even if their full meaning was not clear. I did not sense that there was too

much of a discontinuity with the work of the first week. The patterns gave the students the idea that, like written words in a language, these expressions and operations had meaning.

Furthermore I was teaching my Grade 8 class in a team with two other teachers who were progressing on their own and with a common test coming up, I felt that I had to catch up with them. I thus got my students to write a note with examples on the conventions (things we do) of algebra. Then I gave them an exercise out of the textbook to practise these conventions.

At this stage Jane asked the question: "What is the point of algebra? When will we ever use it?" As Wheeler (1996a, p. 320) says: "But perhaps in no school subject is the teacher so likely to be plagued with questions of the order, 'what's the use of this?'" I tried to explain the usefulness of a symbolic language. I spoke to Jane the following week and asked her how she was feeling when she asked the question. She said that she was feeling frustrated as there was a test and she did not feel well enough prepared for it. I find that this question is often asked when the students are finding the topic difficult.

Lena wrote a note to me in her book that said I was going too fast.

Keshia wrote a whole page:

"Today we did some more algebra. We were writing out the "things we do in algebra". When I was reading over the work, I found that everything was quite simple like multiplying, adding, dividing and subtracting. But I do not wreally (sic) understand why we do not put a letter in front of a number because when you multiply something like  $a \times 4$  why is it not  $a4$  instead of  $4a$  but then in subtraction you can say  $b - 4$ . That is the only problem I had otherwise the lesson was enjoyable because we were all taking part and you were not focusing on people who did not understand and leave others in the learch (sic). Because then I get bored then I start doing my own thing."

Bell, Malone and Taylor (as cited in Kieran, 1992) have commented on this perplexity of students. "beginning algebra students are often perplexed at being permitted to combine  $2a + a + 15$  to  $3a + 15$  but not  $a + a + a \times 2$  to  $3a \times 2$ . As new techniques are learned, students check them against their earlier learning. For example, when they had to simplify  $2x + 3x \times 3$  one girl asked whether BODMAS still applied. Keshia's last comment confirmed what I had sensed was happening in the classroom.

### **Reflection – looking back**

The change in teaching style was not on account of any reading that I had done, but rather arose from the growing conviction that I did not have the skill to keep the class productively occupied. I can still feel the tightening in my stomach that I had at the end of the first week's lessons. For my class the controlled test came too soon for this topic. I should have prepared my own test or persuaded my colleagues to postpone this one to a later date.

### **2.3. The story of week 3: Powers and multiplication. ( two periods - the last 2 maths lessons of the first term)**

I went through the test and summarised the rules of algebra that we had learned so far. I then gave them a worksheet (pp.105 – 106 in appendix 2) which contained exercises of all the types that they would need to know by the end of the chapter in the textbook. It covered adding and subtracting like terms; multiplication; a selection of mixed examples; division; brackets and substitution. The textbook did not have enough basic examples for them to practise. By the end of this week they had covered adding like terms and multiplication. After this they had a week's holiday from school.

#### **The purpose**

To practise the rules and conventions of algebra.

#### **The method**

My style of teaching was to explain a concept to the class using examples which I worked through on the chalkboard. I tried to give the students an example of something which is a particular case of a more general statement. For example, when teaching adding like terms, I would do examples like:  $a + 2a = 3a$ ,  $3xy + 5xy = 8xy$ ,  $4x^2 + x^2 = 5x^2$ . The students would then construct their own generality. They should generalise that like terms are terms that have the same letters and the letters must have the same powers. I encouraged the students to ask any questions, as I needed to know what and how they were thinking. I then gave them some examples to try for themselves. At that stage I walked around the class, helping those that needed

help. The class usually had an exercise to finish for homework. Those that worked fast often had no homework.

### Comments on what happened in the classroom

Rachel struggled with maths and had particular trouble with the meaning of powers. I questioned her about her statement  $x^2 + x^2 = x^6$ . Her explanation was that she added the two x's and the two 2's to get six x's. She explained that  $x^2$  has an x below and another two x's above. She was struggling with the writing conventions.

The most common mistakes made by other students in the test were the following:

2.1  $a + a + a + a = a^4$  was said to be true

2.4  $3p = p \times p \times p$  was said to be true

2.5  $a \times a \div b = \frac{a^2}{b}$  was said to be false.

3.2  $6x + x$  was said to be  $6x^2$

Quite a few of those who said 2.1 was true got 3.1 correct, i.e.  $x + x + x + x + x = 5x$ . Greeno, who is quoted by Kieran (1989, p. 44) in her article on the early learning of algebra, found that beginning algebra students' performance appeared to be quite haphazard for the first while. They made unsystematic errors showing an absence of knowledge of the structural features of algebra.

I like my students to see the big picture of things, so that was why I designed the worksheet after the test. I wanted them to see what they needed to achieve by the end of the chapter. The worksheet elicited two different responses. One said: "This is too much" while Jane said: "Thank you for this. When I learn for the exams, I will mind map this page and not use my work book."

### Reflection – looking back

I do not think what they wrote in the test was a true reflection of what they had learned. It also unsettled some of the students who usually do much better in tests. Because it was a compromise test, common to what all three teachers had taught, some things tested had not been adequately




covered. I felt that my approach had shortchanged some of the girls, as the other classes had not spent as much time on patterns as I had.

**2.4. The story of week 4: consolidation of patterns and procedures. (three periods – the first 3 maths lessons of the term)**

I spent the first three days of this new term consolidating what they had learned in the previous term. They completed a pattern (given in the box below), wrote the rule with letters and in words; they translated into algebra and English; they were reminded of the conventions, how to work with powers, like terms and how to multiply in algebra.

What have we learned:

- Completing a pattern and writing a rule with letters.

					
Number of $\Delta$	1	2	3	4	n
Perimeter	3	4			
- Write the rule in words
- Translate into algebra: divide a number by 3
  - Translate into English:  $2p - 5$
- Conventions:
  - $2 \times n = 2n$      $3 \div p = \frac{3}{p}$
  - $1x = x$
  - $p \times 2 = 2p$     (numbers first)
  - $a \times c \times b = abc$  (letters in alphabetical order)
- Like terms: letters and powers must be the same.

$3x^2 + 4x^2$	yes	$3x^2 + 4x$	no
$2a + b$	no	$2a + a$	yes
- Powers:

$3 \times 3 = 3^2$	and	$p \times p = p^2$
--------------------	-----	--------------------
- Multiplying:

Multiply the numbers.

If the letters are different put them in alphabetical order

If the letters are the same apply the power rules.

$3a \times 4b = 12ab$	$3ab \times 4abc = 12 a^2b^2c$
-----------------------	--------------------------------

Figure 7: A summary of first 3 weeks of algebra. An outline of the consolidation lesson.

They then learned how to divide in algebra.

### **The purpose**

The summary served to remind them of the algebra they had learned during the previous term. It also linked the patterns approach to the generalised arithmetic approach and extended this to division.

### **The method**

I was still using the 'chalk and talk' method. I expected my students to practice examples, but also to find meaning by referring to algebra's likeness to arithmetic.

Comments on what happened in the classroom

Dividing in algebra was a bit of a puzzle to some. They were happy with  $\frac{6}{2} = 3$ , but some were

not so happy with  $\frac{8a^6}{2a^3} = 4a^3$ . A comment was made: "but we are not actually dividing." As a

result in an exercise many students said that  $\frac{6a^5}{a^3}$  could not be simplified.

Having learned how to manage the four operations in algebra, they then practised mixed examples. They had learned to do the parts and now they had to recognise the nature of the problem and take the appropriate action. Many of them were confused by adding and multiplying. They had to remember that unlike terms could not be added but they could be multiplied. Thus  $2a^3 + 5a$  can not be simplified, but  $2a^3 \times 5a$  can be multiplied to get  $10a^4$ . Predictable mistakes were made such as:

$$2x^2 + 3x^2 = 5x^4$$

$$2x^2 \times 3x^3 = 6x^6 \text{ or even } 5x^6$$

As Hewitt (1998, p. 29) points out "If an algebraic structure is used involving an awareness which a student doesn't have, then the only thing left for a student is to memorise a procedure,

and so this procedure is likely to feel as if it is an arbitrary collection of things 'to do'." A reliance on memory often leads to forgetting, or worse, half remembering."

Another common mistake was  $6xy - 3xy = 3$ . Carry, Lewis and Barnard (as cited in Kieran 1992) call this the 'deletion' error. They suggest that students are overgeneralising certain mathematically valid operations.

### **Reflection – looking back**

I read the instruction for teachers in the Just Mathematics textbook (Fitton, et al., 1991, p.58) at a later stage when I was analysing the textbooks. The authors encourage the teacher not to teach the rules, but rather allow the students to discover them for themselves. Students should be encouraged to work from first principles. For example

$$\begin{aligned} 2x^2 \times 3x^3 &= 2 \times x \times x \times 3 \times x \times x \times x \\ &= 2 \times 3 \times x \times x \times x \times x \times x \\ &= 6x^5 \end{aligned}$$

The short way to do it is to write  $2x^2 \times 3x^3 = 6x^5$  where they apply the algorithm: multiply the numbers and add the indices of the same letters.

The authors also say: "The awareness of the rule will develop at widely different stages in different pupils. ... At times you may be discouraged, and may feel that it would be quicker to state a rule and expect the pupils to get on with the job. In the long run such easily acquired rules are just as easily forgotten." In my teaching I showed them the methods using first principles, but I hurriedly moved on to the rules. Maybe many students in my class resorted to memorising the rules.

### 2.5. The story of week 5: removing brackets. (four periods)

They learned how to deal with brackets - the two types were  $(ab)^2 = a^2b^2$  and  $2(a + b) = 2a + 2b$ . Both were taught from first principles.

$$(ab)^2 = ab \times ab = a \times a \times b \times b = a^2b^2$$

$$(2xy^3)^3 = 2xy^3 \times 2xy^3 \times 2xy^3 = 8x^3y^9$$

$2(3 + 1)$  can be simplified in two ways:

$$2(4) = 8 \quad \text{or} \quad 2 \times 3 + 2 \times 1 = 6 + 2 = 8$$

The second method is used if the terms in the bracket are unlike terms.

$$2(x + 1) = 2x + 2$$

Figure 8: Examples of removing brackets.

They also substituted values into expressions. We looked at some more terminology with respect to terms in algebra. They learned the words coefficient, constant, ascending, descending, monomial, binomial, trinomial and polynomial.

#### The purpose

To complete the chapter on basic algebra manipulations, applying the distributive property, justifying the approach by means of arithmetical examples.

#### The method

Following the basic introduction to brackets, I spent most of my time prodding individual students to bring their knowledge of addition and multiplication in this new context.

#### Comments on what happened in the classroom

Mistakes made by Keshia as she tried to understand the topic shows that a wrong turn can be fruitful for developing mathematical reasoning:

"I got number 10 wrong. My answer was  $3x^2 + 2xy$  but the right answer is  $2x^2 + 2xy$  but I will work it out and see where I went wrong.  $2x(x + y)$

makes  $2x^2y$

I did something wrong again.  $2x(y + x) = 2x \times y + x = 2x + (yx)^2$



I need some explanation there because I am confusing (sic) my self.” After an explanation she could see the mistakes that she was making.

Babalwa showed me how she worked out the bracket sums:

$3(a + 4b)$  means that

$3 \times a = 3a$ , and that

$3 \times 4b = 12b$ , so the full answer is

$= 3a + 12b$

Many of the students worked like this. I tried to teach them to do the simple multiplying in their heads as later when the problems were more complicated they would get confused by writing down so many steps.

### **Reflection – looking back**

As the range of their algebra widens, there is an increasing confidence by students to talk themselves through the steps and to identify where they are stuck. This skill is probably of greater importance than getting 100% for a manipulation which is mastered but not grasped.

## **2.6. The story of week 6: algebra consolidation and a test. (three periods and a test period)**

This week was spent in consolidating the ideas that they had learned, as there was the second controlled test on the Friday. This test tested all they had learned about algebra. The following week we looked at Geometry.

### **The purpose**

To revise all the aspects of algebra that had been learned over the past weeks.

### **The method**

Each day I gave them an exercise to do which would cover an aspect of algebra that we had learned. They worked on the exercise and I answered questions. When I was asked many similar questions on a topic I would re-teach that section to the whole class. They also wrote a practice test (p.107 in appendix 2) to see the types of questions they would get and to see where they needed some help. By the time they wrote the test (p.108 in appendix 2) they had completed all the exercises in the textbook as well as all the exercises on the worksheet. (not all in this week)

### **Comments on what happened in the classroom**

This week the focus was on the test. I gave them a summary of what they could expect.

### **Reflection – looking back**

This reflection will focus on the test and the mistakes that were made by the students.

We had spent 6 weeks (about 15 hours) in all on this introduction to algebra. By the end of the time most of them could add and subtract like terms, multiply and divide expressions, remove brackets and substitute values into expressions. The test covered all the aspects that had been taught except for patterns.

Wagner and Kieran (1989) comment that a strong emphasis in arithmetic on finding numerical results of indicated operations contributes to several kinds of problems in algebra. It reinforces an operational rather than a relational interpretation of the equal sign. This is something that students take a long time to outgrow. It also leads to discomfort with expressions like  $x + 3$  as answers to a problem and it contributes to the students' propensity for simplifying algebraic expressions by combining whatever numbers that are available:  $3x + 5 = 8x$  and performing operations just once when more are required:  $5(4x + 3) = 20x + 3$ . This behaviour was shown in the mistakes made in this test. The students were given the expression  $2a + 3b = \dots$  and were asked to choose the correct answer from:  $5ab$  /  $6ab$  /  $2a + 3b$  /  $5a^2b^3$ . 23% of the class chose  $5ab$ . This shows that these students could not accept the lack of closure of an algebraic expression at this stage. When asked to multiply out  $2p(3p + 5q)$  a few of the students gave the answer as  $6p^2 + 5q$  doing exactly what Wagner and Kieran predicted, and despite repeated exercises to practice the contrary.

In the same test they had to state whether the following was true or false:  $\frac{x^3}{x} = 1^3$ , 37% said it was true. Many took the shortcut and cancelled the x's inappropriately. They did not write down the actual meaning of the problem.

When I taught I tried to give the students an example which was a particular case of a more general statement. The students would then construct their own generality. Often students do this well but inappropriately because they would stress aspects that I did not stress and vice versa. In this test they were asked to state whether the following statement was true or false:

$2^3 \times 2^4 = 2^7$  More than half the class stated false. When I went through the test they were upset as they insisted that I *had* taught them that  $2^3 \times 2^4 = 4^7$ . After some discussion I realised that they had heard me saying, "multiply the numbers and add the indices (of the letters)" for the example  $2a^3 \times 5a^2 = 10a^5$  and had inappropriately applied it to an example with no letters. On that same point, a very common mistake made by students in Grade 9 to Grade 12 is  $(a + b)^2 = a^2 + b^2$ . Some of my more able Grade 9 students argued that I had taught it to them in Grade 8. When I showed them that  $(ab)^2 = a^2b^2$  they agreed that was what they had learned the year before. They had made an incorrect generalisation from a multiplication context to an addition context.

In a study with Grade 10 students, Wong (1997) found that the error of multiplying the bases was frequently made. It did not matter whether the indices were letters or numbers. e.g.

$3^x \times 3^y = 9^{x+y}$ . When asked about it the student replied: "3 times 3 gives 9. x and y could not be multiplied together, so y should be added to x." The same students were able to do  $x^5 \times x^3 = x^8$  quite happily. When another student was asked why he wrote  $53^x \times 53^y = 53^{x+y}$  but then wrote  $3^x \times 3^y = 9^{x+y}$ , his reply was that he had written down the 9 automatically. The  $3 \times 3 = 9$  rule is powerful.

## **2.7. The story of weeks 7 - 10: Algebra with negative coefficients. (13 periods and a test period)**

After a break of a few months I came back to algebra by looking at integers. The focus was on how to add, subtract, multiply and divide negative numbers. The students tried very hard to construct rules or generalisations for themselves. I got statements like: "When you have two minus signs you add the numbers and when the signs are different you subtract the numbers..." Said by a student who works hard to get it right, but finds it difficult to be consistent. They looked for patterns or rules to make sense of the work. At the same time they were revising those procedures with algebra too. In each case we did a few examples with numbers and then practised the same procedures with letters. The focus of this chapter was on the negative numbers and sometimes the algebra knowledge was put on hold. Directed numbers is another difficult concept for Grade 8's to grasp. Combining it with algebra is particularly confusing as the new integer procedures are introduced before the algebra has been learned fully.

### **Comments on what happened in the classroom**

Tony stated that  $2x - 1 = 1$ . She was so focused on getting the signs of the numbers right that she did not 'see' the  $x$ . When questioned about it she said: "Two take away one is one."

Teacher: "What about the  $x$ ?" Silence for a while, and then she said, "You just ignore it."

When questioned about  $2x + 1$  she immediately replied  $3x$ . I explained adding and subtracting like terms to her again. In the end of year examinations she again said that  $2x + 1 = 3x$ .

The question was to simplify  $-3x - (x + 3)$

Jane said the answer was 0. Her page had a bit of working on it with crossings out. She had the correct working:  $-3x - x + 3$  but then she crossed it out. Her explanation was that she hadn't 'done anything'; meaning that she did not think that she had done any subtraction, so that could not be the answer. She eventually decided that

$x+3 = 3x$  and so  $-3x - 3x = 0$ .

Susan and Keshia made similar mistakes with removing brackets. The question:

$8 - 3(x + 5)$  became  $8 - 3x + 2$ .

On the side of her page, Susan had written:  $-3 \times x = -3x$  and  $-3 + 5 = 2$  She added the last terms instead of multiplying them.

The question:  $-3(2x - 1)$  became  $-6x + 4$ . Keshia also added the last terms, but changed the sign, as she must have thought that two minuses make a plus.

### **Reflection – looking back**

I was dissatisfied with the double confusion: algebra and negative numbers. I decided to teach negative numbers first with the following year's Grade 8 group so that the integer arithmetic would be available before learning the algebra using negative coefficients.

### **2.8. The story of week 11: equations. (four periods)**

The final week was spent solving equations. The lessons on equations went much better than I expected they would. The students found that they could do the algebraic manipulation required by equations. The exercises presented had no exponents to worry about. We started with examples that they could solve by inspection.

$$2x + 3 = 7.$$

We then went on to equations which had the unknown on both sides. This type with an  $x$  on both sides of the equal sign is known to be conceptually much more difficult.

$$2x - 11 = 7 - x$$

and then on to equations with brackets.

$$3(x - 3) - (2x - 1) = 4(2x + 5)$$

### **The purpose**

To gain the skills to be able to solve simple equations.

### **The method**

The textbook showed both the balancing and the transposing approaches to solving equations. As I had read (Kieran, 1989) that beginning algebra students do not see these two commonly used approaches as equivalent methods, I decided to focus on one method only. The balancing approach treats the equation as a logical equality statement with equivalent expressions on either side of the equality sign. The argument of this balancing approach is that applying the same

operation to either side preserves the equivalence. Using this approach is supposed to enhance meaning. Typically it is verbalised as 'what you do to one side you must do to the other.' On the other hand, the transposition method presents an economical manipulative procedure. I therefore decided to teach solution of equations by the transposition method as I was pressed for time. I started with solving simple equations by inspection. For example  $2x + 3 = 7$ . I then asked the students to explain what they were doing in their minds. Many of them said that they first subtracted 3 from 7 and then divided the answer by 2. We called this the 'undoing method' and wrote down the steps. I used this as my justification for using the method of transposition.

balancing

$3x + 2 = x - 1$   
 $3x + 2 - 2 = x - 1 - 2$       subtract 2 from both sides  
 $3x = x - 3$       simplify  
 $3x - x = x - x - 3$       subtract x from both sides  
 $2x = -3$       simplify  
 $\frac{2x}{2} = \frac{-3}{2}$       divide both sides by 2  
 $x = \frac{-3}{2}$       simplify

transposition

$3x + 2 = x - 1$   
 $3x = x - 1 - 2$       take 2 to the right hand side and subtract  
 $3x = x - 3$       simplify  
 $3x - x = -3$       take x to the left hand side and subtract  
 $2x = -3$       simplify  
 $x = \frac{-3}{2}$       divide right hand side by 2

Figure 9: Two methods of solving equations

### Comments on what happened in the classroom

I spent one week on solving equations. I did not solve any "problem sums" at this stage. In looking at the mistakes made at the end of this teaching session and at the end of the year, I felt that the majority of students had mastered this skill. There were four equation questions totaling twelve marks in the final examination. The median of the scores was nine out of twelve. The most common mistake made was in removing brackets. One question was:

$$5(x + 1) - (2x + 1) = 2(x - 2)$$

$5x + 1 - 2x + 1 = 2x - 2$       The error was made by not multiplying the second term in each bracket. 5 students made this mistake.

$5x - 2x - 2x = -1 + 1 - 2$       There was also the error of not changing the sign of the second 1

$$x = -2$$

The following three responses were given in answer to the question:

Solve for x:  $5(x - 6) + 11 = 7(x + 1)$  This became:

- $5x - 30 + 11 = 7x + 1$       Here the student multiplied the first bracket correctly but not the second one
- $5x - 11 + 11 = 7x + 7$       The student 'added' the numbers in the first bracket but multiplied the second bracket correctly
- $5x - 30 + 11 = 7x - 7$       This may have been a slip with the incorrect sign on the right hand side. An interview with the student would have clarified the situation.

Another equation gave some insight into the students' reasoning. The question was:

Solve for x:  $(x - 3) - (x - 2) - (x - 1) = 0$

$$x - 3 - x + 2 - x + 1 = 0$$

$$x - x - x = 3 - 2 - 1 \quad *$$

$$-x = 0$$

$$x = 0$$

Four able students got to \* correctly, but did not get it right from there. Two of them reduced the left hand side to  $-3x$  and the other two reduced it to  $-2x$ . Yet in the same test they all correctly simplified  $2x - x - 2x$  to  $-x$ . The first situation could be more difficult for the student because there is no numerical coefficient to be seen in front of the  $x$  terms. This case might be made easier by an additional step of inserting coefficients  $(1x - 1x - 1x)$  before simplifying to  $-1x$  or just  $-x$ .

Another student used a trial and error approach, but did not get past the first trial.

She let  $x = 1$ .  $(-2) - (-1) - (0) = 0$

In that same question another student solved the problem like this:

$$(x - 3) - (x - 2) - (x - 1) = 0$$

$$\begin{array}{rcl} -2x & -x & - 0 = 0 \end{array}$$

$$\begin{array}{rcl} -3x & & = 0 \end{array}$$

$$\begin{array}{rcl} x & & = 0 \end{array}$$

The student arrived at the correct answer, but with the wrong process. She was still at the operational stage of working with numbers where she needed a single answer for  $x - 3$ .

### Reflection – looking back

The test of whether the students have learned how to solve equations will come in the next few years as they learn to do more with equations. At the moment they can solve the basic equations by following the rules of transposing. As soon as something takes the focus off the actual process, their weaknesses, if any, will show up. I will not use this transposing approach next year as it does not foreground the idea of the equal sign being the pivot, the linking of two equal expressions. I can however anticipate the difficulties that I will face with the lengthier balancing approach. One of these will be that the students are inclined to write one change per line resulting in unnecessary time being wasted. Menghini (1994, p.12) says that “ many students



seem to prefer long, monotonous, obviously repetitive processes, which, as they become automatic, require very little concentration or reasoning (but are guaranteed) instead of brief, concise processes with few calculation which, however, require active distinguishing of similarities and differences, and understanding the rules, and the ability to synthesize.”

Susan confirmed Greeno's statement (quoted in Kieran, 1989) about beginning algebra students making unsystematic errors. She made the following series of mistakes:

6.2	$y^2 + y^2 = y^4$	×
6.4	$3y^5 \times 3y^5 = 9y^5$	×
6.5	$4x \times 8x^3 = 32x^4$	✓
6.7	$-2(x+2) + (x-4) + 3$	
	$-2x - 4 + x - 4 + 3$	✓
	$-6x + x - 4 + 3$	×
	$-6x^2 - 4 + 3$	×
	$-10x^2 + 3$	×
	$-7x^2$	×

Figure 10: Susan’s mistakes

In the end of year examination the students were asked to simplify  $\frac{2a + 3a}{5a} - (2a - a)$

Jane, in the following excerpt, focused on the algorithm of dividing instead of looking at the problem carefully.

$$6.9 \quad \frac{2a+3a}{5a} - (2a-a)$$

$$\frac{3a}{5a} =$$

$$\frac{1}{3a} + \frac{1}{2a} = 5a - a = 4a \quad \times \quad \bigcirc$$

Figure 11: Jane's attempt to simplify the problem

In the same problem two students simplified the first term to  $\frac{5a}{5a}$  and then made that equal to 0.

Another five students also wrote down  $\frac{5a}{5a}$  but they simplified it to a.

The point is that when operations are combined as here with addition, subtraction and division, the beguiling  $\frac{5a}{5a}$  leads some to subtract giving 0 and others to apply a known rule for  $\frac{5}{5}$  making

it 1 but not following this exact step from arithmetic with regard to  $\frac{a}{a}$  which by magic becomes

a. In the final test many students said that  $y^2 + y^2 = y^4$ .

In the concluding remarks of her article on the early learning of algebra, Kieran (1992) identifies the principal difficulties that learners struggle with as: the meaning of letters, the shift to a set of conventions different to those of arithmetic, and the recognition and use of structure. Some students do not have the ability to 'see' the surface structure of algebraic expressions containing various combinations of operations and literal terms. This problem seems to continue throughout

the algebra career of many students. (As seen when simplifying  $\frac{a+b+c}{a+b} = c$  for example)

## 2.9. Conclusion

I feel that the majority of my students have managed to assimilate a fair amount of algebraic knowledge during 2002. When one reads of all the difficulties of learning algebra, I think that many have coped well, if coping well means passing the end of year examination. (pp.109 - 114 in appendix 2) But as Skemp (1971, p. 35) warns: "Even those that get off to a good start may through absence, inattention, failure to keep up with the pace of the class, or other reason, fail to form the concepts of some particular stage. In that case, all subsequent concepts dependent on these may never be understood, and the pupil becomes steadily more out of his depth."

I will never know whether my Grade 8 students might have had a greater understanding of algebra if I had approached algebra in a different way. The pitfalls of learning algebra are many - as Wheeler (1996b, p. 149) says: "Learning is never additive and almost never convergent. It coalesces and disintegrates, it continually backtracks and recasts itself, as if it were some hugely-complex, on-going experiment that never concludes." As this was the students' first exposure to formal algebra, more learning could take place in their Grade 9 year where they carry on learning algebra. As my Grade 8 class of 2002 was aware of my research, we often discussed their process of learning. Some of them wrote their comments on paper, while others spoke to me in the classroom. This articulation of their problems during their learning might have helped their learning process even though I was not aware of it at the time.

From the above account of my teaching, I can see that I viewed algebra mainly as a generalisation process. As I presented it to my class, this generalising had two aspects. The first involved generalising a pattern by writing a rule, the second aspect required generalising what was done in ordinary arithmetic, only this time by using letters. This meant that my students had to learn the conventions and rules and be able to apply them. They had to practise the skills by doing many examples. I found myself asking, "Is this what algebra is or is algebra something that is far broader?" It was the quest for the answer to this question that led me deeper into the literature.

### CHAPTER 3      A BRIEF HISTORY OF ALGEBRAIC NOTATION AND SOME DEFINITIONS OF ALGEBRA.

I was seeking an answer to the question “What is this branch of mathematics called algebra and where does it come from?” As I saw the way in which a standard notation for algebra developed, I became aware of the advantages of a tight symbolic notation. This reflects the essentially abstract nature of algebra.

The word *algebra* is the Latin variation of the Arabic word *al-jabr*. This word comes from the title of the book *Hisab al-jabr w'al-muqabalah* written by Mahammed ibn-Musa al-Khowarizmi in Baghdad in about 825 AD, which dealt with solving equations (Baumgart 1969). The word *al-gabr* means *restoration* or *completion* and seems to refer to the transposition of a subtracted term to the other side of an equation. The book covers numerous problems stated in words including quadratic equations. It was this elementary work, which was translated into Latin, that has had a marked effect on school algebra up to the present (Boyer, 1968).

#### **The history of algebraic notation:**

In ancient texts of mathematical problems the description of the problem was often given in awkward language with few symbols. This gradually changed. As time went on the process of generalising became easier as algebra became less wordy and more symbolic. By the time of Descartes a recognisably modern notation had become standard. In our present use of algebra, the depiction of a problem is compact and the solution is consequently much easier to work out.

Kieran (1992) identified three stages through which algebra has evolved. The first stage is called the rhetorical stage and belongs to the period before Diophantus (c. 250 A.D.). It is characterised by the use of ordinary language descriptions for solving problems. No symbols or signs were used for the unknowns. To illustrate this I use an example from the Arab mathematician al-Khowarizmi:

Question: What must be the amount of a square, which, when twenty-one dirhems are added to it, becomes equal to the equivalent of ten roots of that square?

Solution: Halve the number of the roots; the half is five. Multiply this by itself; the product is twenty-five. Subtract from this the twenty-one which are connected with the square; the

remainder is four. Extract its root; it is two. Subtract this from half of the roots, which is five; the remainder is three. This is the root of the square which you required and the square is nine. Or you may add the root to half of the roots; the sum is seven; this is the root of the square which you required and the square is forty-nine.

In modern terms he is solving the equation:  $x^2 + 21 = 10x$

The solution is  $x = \frac{10}{2} \pm \sqrt{\left(\frac{10}{2}\right)^2 - 21} = 5 \pm \sqrt{4} = 3 \text{ or } 7$

(Cummins, 1969, p. 260)

The second stage, called the syncopated or intermediate stage, lasted for over 1200 years, from the time of Diophantus (c. 250 A.D.) until Vieta (1540). This stage was initiated by Diophantus who made a start toward modern symbolism by introducing abbreviated words. He also used abbreviations for powers of numbers and for relationships and operations. The main concern of the algebraists of that time was to discover the value of the unknown(s) rather than expressing generality. This syncopated symbolism did not develop to any great extent over the 1200 years.

The polynomial  $x^3 - 5x^2 + 8x - 1$  is written  $K^{\epsilon}\alpha\zeta\eta \quad \Delta^{\epsilon}\epsilon M^{\circ}\alpha$  (cube 1 unknown 8 minus squares 5 units 1). The numbers are represented by the small Greek letters.

François Viète or Vieta is identified as initiating the third stage which is called the symbolic stage. Vieta was influenced by reading a Latin translation of Diophantus' work. He decided to use the capital letters of the alphabet to represent all the quantities in the calculations whether the values were known or not. The vowels were used for unknown quantities and the consonants for known quantities. He adopted the Germanic symbols for addition, +, and subtraction, -, but the remainder of his work consisted of words and abbreviations. The second power was a quadratus, multiplication was signified by the word 'in' and for equality he used an abbreviation of the word aequalis. He showed an awareness of the relationship between the roots and the coefficients of a polynomial equation. It was now possible to express general solutions to

equations. This was the start of the symbolic language that we know today. (Dedron & Itard, 1959).

The equation  $3x^2 - 5x + 6 = 0$  would be written  
3 in A quad - 5 in A plano + 6 aequatur 0

Descartes used letters that are very similar to our modern notation. He replaced all uppercase letters with lowercase ones and he used the last three letters of the alphabet as unknowns.

The following example shows the development of the ways of writing a particular quadratic equation over the span of two centuries (Hogben, 1937, p. 303):

Regiomontanus, A.D. 1464:  
3 Census et 6 demptis 5 rebus aequatur zero

Pacioli, A. D. 1494:  
3 Census p 6 de 5 rebus ae 0

Vieta, A.D. 1591:  
3 in A quad - 5 in A plano + 6 aequatur 0

Stevinus, A.D. 1585:  
 $3 \textcircled{2} - 5 \textcircled{1} + 6 \textcircled{\cdot} = 0$

Descartes, A.D. 1637:  
 $3x^2 - 5x + 6 = 0$

From the above description algebra seems to require mainly the ability to write a problem in a concise notation in order to make it easier to solve the problem. Algebra however has developed into something far broader than that. As Baumgart (1969, p. 234) says: “A satisfactory definition

requires a two-phase approach: (1) Early (elementary) algebra is the study of equations and methods of solving them. (2) Modern (abstract) algebra is the study of mathematical structures such as groups, rings and fields.” The following definitions make this explicit.

### Definitions

There are many definitions of algebra. Very often the definition depends on the context in which it is written. The Chambers Twentieth Century Dictionary definition is a general definition:

“Algebra is a method of calculating with symbols - by means of letters employed to represent quantities, and signs to represent their relations, thus forming a kind of generalised arithmetic; any of a number of systems using symbols and involving reasoning about relationships and operations.” (Macdonald, 1972, p. 30) School textbooks often have a simple definition that indicates the approach taken. For example “Algebra is a language for mathematics that allows one to generalise, to show structure and to communicate in a precise and unambiguous way.” (Maths Education Project 2000b, p. 55). The emphasis here would be on generalising. The New South African Curriculum defines algebra as the language for investigating and communicating most of Mathematics. Algebra can be seen as generalised arithmetic, and can be extended to the study of functions and other relationships between variables. (Revised National Curriculum Statement, 2002)

Brown and Coles (1999, p. 155) quote a broad definition which covers most interpretations.

“Algebraic activity includes:

1. Generational activities which involve generalising from arithmetic, generalising from patterns and sequences, generating symbolic expressions and equations which represent quantitative situations, generating expressions of the rules governing numerical relationships.
2. Transformational activities which involve: manipulating and simplifying algebraic expressions to include collecting like terms, factorising, working with inverse operations, solving equations and inequalities with an emphasis on the notions of equations as independent 'objects' which could themselves be manipulated, working with the

unknown, shifting between different representations of, function, including tabular, graphical and symbolic.

3. Global, meta-level activities which involve: awareness of mathematical structure, awareness of constraints of the problem situation, anticipation and working backwards, problem solving, explaining and justifying.”

Hewitt (1998) agrees with the third aspect when he says: “whereas arithmetic is concerned with answers, algebra is concerned with operations and structure. Algebra is concerned with awareness of awareness.”

### **Interlude for reflection – my own initial definition**

In my teaching I introduced the symbolic language in the first week of learning algebra. To the beginning algebra student this formal symbolic notation is not a natural means of representing a problem. The ancients took 300 years to come to a consensus on symbols that are universal and concise. My students had to learn it in one day.

At the start of 2002, before I started teaching algebra from the perspective of researching it, I wrote down my own definition of algebra: “Algebra is a tool for generalising arithmetic which can then be used for solving a particular problem.” As such it was arithmetic with letters, in that we were doing similar operations in algebra to those of arithmetic. Then we used this letter arithmetic to write equations and do manipulations on the equations to get a solution. Brown’s definition of algebra has three aspects to it. In my teaching I only touched on two of them: generalizing from patterns and manipulating expressions (transforming). I do not think that I explicitly made my students aware of mathematical structure from a global point of view. While I was introducing algebra I was not focusing on meta-level activities because at that stage I was emphasising the conventions of the manipulations. Any awareness that they developed of the structure would have been a bonus. My purpose was to make them familiar with the skills through practice.



Perhaps, in my future teaching of algebra, I need to introduce algebra in another way so that my students could develop a broader view of algebra.

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## **CHAPTER 4      APPROACHES TO TEACHING ALGEBRA**

The teaching and learning of algebra has attracted a great deal of research interest. Various approaches aimed at making learning meaningful for students have been proposed. There are many definitions of algebra each of which is related to different conceptions of algebra and leads to different ways of introducing it.

### **4.1 Introduction to four approaches to teaching algebra.**

In my search through a variety of books and journals outlining different approaches to introducing algebra to students I came across a book which synthesised the various positions. The book was a result of an international colloquium on “Research perspectives on the emergence and development of algebraic thought,” in Montreal in 1993. In the book (edited by Bednarz, Kieran & Lee, 1996) four approaches to introducing algebra are examined. Each has its supporters who write convincingly. The different initial approaches largely determine the early algebraic conceptions that students develop. The students hold onto these conceptions which then interfere with the processes of further algebraic knowledge construction.

The four approaches suggested in the book are:

1. Generalisation of numerical and geometric patterns and the laws governing numerical relations.
2. Problem solving
3. Modeling of physical phenomena
4. Focusing on the concepts of variable and function

A good algebraist needs each of the above concepts of algebra. A student of algebra should be able to use algebra to solve problems, to model situations, to handle functions, and to make generalisations. The question is where to begin in learning algebra. Choosing one of these as a starting point will affect how the others can be reached. The consideration of how each approach includes or links to the others is important. Each approach includes aspects of the other three approaches. The difference is the emphasis made while teaching.

I am going to discuss each approach in detail, comment on the related definition of algebra and look at the consistent mistakes that the students might make as a result of that approach.

4.2. Generalisation

The definition used by this approach is that algebra is a language with which one can communicate abstract and complex ideas. In this view algebra is especially suited to expressing generalities. It has its own set of rules which need to be learned and practised. By narrowing the focus of generalisation to generalised arithmetic the definition can be adapted so that algebra is presented as the meta-language of arithmetic. Through the language of algebra students are encouraged to express generalities and to express relationships as formulae.

Generalising is a natural process and we do it easily if not accurately. For example "All birds fly" But it is also a difficult process. To generalise in the mathematics classroom, the students need to see the generality of the problem they are dealing with and then express it in algebraic language.

An example from the textbook On Track with Maths (Fitton S., De Jager C. and Blake P., 2000, p.57) shows how algebra explains in general why a certain number play always works:

English	Shorthand
Think of a number	$x$
Add 3	$x + 3$
Double the result	$x + x + 3 + 3 = 2x + 6$
Subtract 4	$2x + 6 - 4 = 2x + 2$
Divide the result by 2	$x + 1$
Subtract the number you first thought of.	$1$

Figure 12: Example of shorthand

In teaching, the teacher often gives the students an example of something which is a particular case of a more general statement. The students then construct their own generality. Often students do this well but inappropriately because they stress aspects that the teacher did not stress and vice versa. For example, there was an outcry in the classroom when I told them that the answer to the problem of multiplying out the brackets  $(a + b)^2$  was  $a^2 + 2ab + b^2$ . They said that I had definitely taught them that the answer should be  $a^2 + b^2$ . After a discussion we realised that what they were remembering was that  $(ab)^2 = a^2b^2$ . They had focused on the rule that each thing in the bracket must be raised to the power and not the fact that this rule only applies with products and quotients and not sums and differences.

Mason's (1996, p. 65) conjecture is that "when awareness of generality permeates the classroom, algebra will cease to be a watershed for most people." Most professional mathematicians take generalisation for granted as they are so used to it. The shift that took place in their minds, which they are probably not aware of any more, is the precise point of difficulty for novices.

Generalisation requires that teachers think and teach mathematically. Developing mathematical thinking requires teachers to get their students to work at expressing their own generalisations. Algebra is the language in which generality is expressed. Mason (1996, p.71) goes on to say that to teach algebra successfully "the presence of someone whose attention is differently structured, whose awareness is broader and multiply-leveled, who can direct or attract student attention appropriately to important features, is essential." Teachers need to be aware of the problems in learning algebra so that they can anticipate the problems and use a variety of examples to highlight these issues. The processes involved in generalising could be summarised as detecting sameness and difference, making distinctions, repeating and ordering, classifying and labeling. They are the basis of mathematical thinking and more specifically of algebraic thinking.

A more focused aspect of generalisation, is that algebra is generalised arithmetic. Algebra is seen as an extension of arithmetic. The rules of algebra are seen as a symbolic statement about the rules of arithmetic. It is logical as it is seen to be building on to knowledge that the students know. Below is an example from the textbook Outcomes-based Classroom Mathematics (Laridon et al., 2000a, p.67)

#### Activity 4.8

- Work in pairs
- Copy and complete the addition patterns in question 1 by writing down the correct answers
- Try to discover a general rule

1 a)  $2 + 2 = \underline{\quad} = 2 \times \underline{\quad}$

b)  $3 + 3 = \underline{\quad} = 2 \times \underline{\quad}$

c)  $4 + 4 = \underline{\quad} = 2 \times \underline{\quad}$

d)  $10 + 10 = \underline{\quad} = 2 \times \underline{\quad}$

e)  $100 + 100 = \underline{\quad} = 2 \times \underline{\quad}$

f)  $n + n = \underline{\quad} = 2 \times \underline{\quad}$

g)  $3x + 3x = \underline{\quad} = 2 \times \underline{\quad}$

h)  $7y + 7y = \underline{\quad} = 2 \times \underline{\quad}$

Figure 13: Example of generalised arithmetic

Wagner and Kieran (1989) suggest that generalised arithmetic can be thought of from two points of view: as generalised *arithmetic* or as *generalised* arithmetic. In the first, the focus is on the link between algebra and its numerical referents and in the second, the focus is on the structural aspects of the number system. Much of the emphasis in junior school arithmetic is on finding the answer. In algebra they are now required to recognise and use the structure that they have been able to overlook in arithmetic.

The success of a generalised arithmetic approach depends on developing a global awareness of the way numbers ‘work’ as a system. It highlights opportunities to initiate discussion about properties of numbers, to become aware that there are rules to arithmetic. Arithmetic of small numbers is so automatic to students that they often fail to see what the question is about when we ask questions about numbers. As they become more articulate about numbers and their properties, they will be laying the groundwork for algebra. Questions about rules of arithmetic, order of calculation and so on should be touched on often in the mathematics classroom. It should start right from the early stages in the student’s schooling.

Linchevski (2002) gives us a practical suggestion on how this might be done. Algebra is generalised arithmetic. The algebraic structure draws legitimisation and meaning from the rules in the world of numbers. This motivates the teaching approach at the junior school level to teach arithmetic for algebraic purposes. When numbers are taught in junior schools, the children are often encouraged to look for number combinations to make the calculations easier. For example:  $97 + 256 + 3$  they are encouraged to add the 97 and 3 first. In  $\frac{1}{3} + \frac{4}{5} + \frac{2}{3}$  they are encouraged to add the first and last terms. These examples encourage looking for short cuts, rather than keeping the rules. These must be balanced with examples that encourage structure.  $217 + 63 - 105 + 42$  is a numerical prototype of an algebraic expression. In arithmetic you go from left to right, but in algebra you need to jump to add the like terms. Therefore ask them to do the 3-digit numbers first and then the 2-digit numbers. They might not be sure whether the - sign goes with the 42 or not.  $63 - 42$  or  $63 + 42$ ? According to Linchevski there is no research that systematically investigates whether structural understanding in numerical contexts would help with algebra. The question whether systematic work on structure sense within the world of numbers will lead the students to a better understanding of algebra is still to be answered.

Algebra emerges when students become aware of the awarenesses that allow them to operate on objects. Thinking about the properties of numbers is one way of standing back from engagement in the particular and becoming aware of the processes. Algebra awareness requires shifts of attention, which make it possible to be flexible in seeing a group of written mathematical symbols such as  $x + 3$  on the one hand as an expression and as a value and as an object and as a process on the other. In a task where the student has to write a number "three more than  $x$ " many beginning algebra students have difficulty in considering  $x + 3$  as an answer. To be able to work in algebra students need to change their expectations that answers need to be numbers as in arithmetic. They need to see an 'answer' as an expression and as a value. They also need to see the  $x + 3$  as an object, which can be worked on, and a process (add three to the variable). Variables in this approach are seen as pattern generalisers and the algebraic skills are centered on translating and generalising known relationships among numbers.

- Complete the table

Number of cows	1	2	3	4	10	n
Number of legs	4	8				
- Write the relationship between the number of cow and the number of legs in words.

Figure 14: Example of generalising a pattern

To move from the theoretical aspects of this discourse to the more practical, Mason (1996, p. 81) suggests a framework in which to work. He indicates three key actions: manipulating, getting-a-sense-of and articulating.

- “manipulation (whether of mental, physical or symbolic objects) provides the basis for getting a sense of patterns, relationships and generalities, and so on; (In the above case, completing the table.)
- the struggle to bring these to articulation is an ongoing one, and that as articulation develops, sense-of also changes; (Writing the relationship in words.)
- as you become articulate, your relationship with the ideas changes; you experience an actual shift in the way you see things, that is, a shift in the form and structure of your attention; what was previously abstract becomes increasingly, confidently manipulable. " (Fill in the n column in the above table.)

He suggests that teachers use them in a spiral, going back up the spiral when a difficulty is encountered. In the classroom one would encourage the students to ‘see’ the problem, then to describe it in words and then record it in algebraic language. The process of verbalising helps the student to clarify her thoughts and meanings. Class discussions are particularly rich in enabling the teacher to be aware of and to guide the students’ understandings.

- Complete the table

Number of cows	1	2	3	4	10	n
Number of legs	4	8	12	16	40	4n
- Write the relationship between the number of cow and the number of legs in words

The number of legs is four times the number of cows.

Figure 15: Solution to figure 14

Mason's comment on this approach is that "significant discordance was found between arithmetic and algebra: Different rules applied, and answers obtained in one domain did not have to agree with those obtained in the other. They show that the superficial reliance on algebra as generalised arithmetic, as letter-arithmetic is in itself insufficient to obtain mastery and smooth transition." (Mason 1996, p. 78) A study by Chalhoun and Herscovics (Kieran, 1989, p. 43) showed that students *do* develop an awareness that the conventions used in algebra are different from those used in arithmetic. Wheeler (1996b, p. 148) does not consider a "smooth transition" necessary to learning algebra. He asks: "Why should we not try to teach algebra to secondary students as something totally fresh and different from anything they have learned before?" He claims that students need to work through developmental "cuts", "ruptures" and intellectual "conflicts." Learning will not take place effectively (or even not at all) if the difficulties are removed. He says that teachers need to work out a compromise between smoothness and necessary roughness.

The particular problems that arise from this approach:

- Some children are strong patterners. They see patterns quickly and are very unwilling to do any thinking about the meaning of the processes. They don't see the need for understanding because they can just do what is required from having picked up the pattern. The mistakes these children make arise from going to the algebra too quickly without thinking. They make unwarranted assumptions and end up writing down incorrect statements.
- The patterns that they spot are trivial.

For example, the students were given the problem of looking at numbers that have an integer as the result of subtracting 1, then dividing by 4, and then multiplying by 3. One student said that all the answers were multiples of three. When asked whether that was always true, he went off to check. Was it effective for the student to do lots more calculations until he realised that if he multiplied a number by three the answer must be a multiple of three?



N	1	5	9	13	17
$\frac{n-1}{4} \times 3$	$\frac{1-1}{4} \times 3$	$\frac{5-1}{4} \times 3$	$\frac{9-1}{4} \times 3$	$\frac{13-1}{4} \times 3$	$\frac{17-1}{4} \times 3$
Answer	0	3	6	9	12

Figure 16: The pattern

I encountered a situation illustrating this:

The class was given this exercise from the textbook, Just Mathematics 6. (Fitton et al., 1991, p. 51):

**Worksheet 4**

1. Write down and complete this number pattern until you have 6 rows  
Can you see the rule?  
Write it out in words.

1 + 2 + 3 = 6  
2 + 3 + 4 = 9  
3 + 4 + 5 =

2. If three consecutive numbers are added and give the answer 369 what are the three numbers?

Figure 17: First part of a pattern worksheet

The answer given by some students was “you take the middle number of the previous row and add the next two numbers to it.” While this is valid, it does not easily lead to the algebra of the pattern. (A richer answer could have been that the sum of three consecutive integers was a multiple of three or better still that the sum of three consecutive integers was three times the middle number)

- The students are happy with a formula that works for a few cases. There is little sense of obtaining a generally valid formula. That is, the interest to find a formula which works for all cases, is not present. In the Grade 8 June exam 2001 the students had to find a formula linking the number of edges, faces and sides of the regular polyhedra.

Polyhedron	Number of faces: <b>f</b>	Number of edges: <b>e</b>	Number of vertices: <b>v</b>
<b>Tetrahedron</b>	4	6	4
<b>Cube</b>	6	12	8
<b>Octahedron</b>	8	12	6
<b>Dodecahedron</b>	12	30	20
<b>Icosahedron</b>	20	30	12

Write down a formula which links all three f,e, and v together.

Figure 18: Completed table

Some answers were:  $e + f + v = 14$     and     $f \times e \times v = 96$

- Focusing on the particular only.

The example given by Mason involved ‘arithmagons’.

The students were given a particular set of numbers on the edges and asked to investigate the solution for the numbers which would fit in the corner positions such that their sum would equal the edge number.

(Mason, 1996, p.78) His comment is that most students struggled merely to find one set of numbers that was a solution.

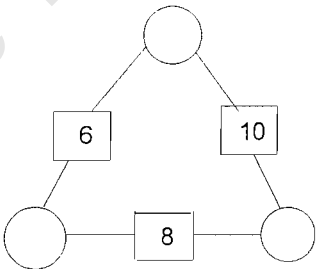


Figure 19: Arithmagon

- Incorrect generality.

Every teacher has seen a rule applied without reference to its context. e.g. “Two minuses make a plus.”

$$-3 \times -5 = 15 \quad \text{but}$$

$$-3 - 5 = 8$$

Another very common error of this type is that  $15x - 4$  becomes  $11x$  or  $4abc - 4ab$  becomes  $c$ .

These are some examples of generalisations that students make as they learn the subject. As mentioned before they often make incorrect generalisations because their minds are emphasising the wrong things.

- A strong emphasis in arithmetic on finding the numerical results of indicated operations contribute to several kinds of problems in algebra. It reinforces an operational rather than a relational interpretation of the equal sign. This is something that students take a long time to outgrow. It also leads to discomfort with expressions like  $x + 3$  as answers to a problem and it contributes to the students propensity for simplifying algebraic expressions by combining whatever numbers that are available ( $3x + 5 = 8x$ ) and performing operations just once ( $5(4x + 3) = 20x + 3$ ).

#### 4.3. Problem solving

An exploration of problems can form the basis of an initial algebra course. Problem solving involves processes such as forming and solving equations, generalisation, working with functions and formulas. An open exploration of problems can emphasise forming and solving equations. Wheeler (1996a) comments that problem solving connects algebra to what some people regard as the main purpose of mathematics – the solving of well-formulated problems.

The definition of algebra that is used by this approach is summarised by Wheeler (1996a, p. 319).

- Algebra is a symbolic system--symbols are the signs by which we often recognise the presence of algebra--but it's also more than a symbolic system.
- Algebra is a calculus--among its primary elementary uses is the computation of numerical solutions to problems--but it's also more than a calculus.
- Algebra is a representational system--it usually plays a large role in the mathematization of situations and experiences-- but it's also more than a representational system.

Problems are usually seen as word sums. I have identified three types of problems given in the textbooks:

1. Real- life problems: Problems that people solve in the ordinary course of their lives. For example: Which taxi is cheaper? Patel taxis at 80c plus 25c per minute or Limo taxis at R1,20 plus 20c per minute. (Human et al., 2000a, p. 213)
2. Contrived real-life problems: Problems that look like real life problems but they are simplified. Either the numbers are made easier to work with or the context is simplified. For example: An ambulance is 100km behind a car. The car travels at 60km/h and the ambulance at 80km/h. Will the ambulance overtake the car? Explain your answer. (Maths Education Project 2000a, p.12)
3. Mathematical problems: Problems that are usually found in traditional mathematics textbooks and in mathematics competitions. For example: If you multiply a certain number by 3 and then add 4, you get 19. Find the number. (Fitton et al., 1991, p.85)

The problem solving approach is very rich in providing a context for thinking mathematically and for using the skills that one has learnt. But how does one learn the skills in the first place? The learning of some manipulative skills needs to be fitted in at some stage. There is not much in the literature dealing with practical ways of introducing algebra for this approach. The jump from arithmetic to algebra is difficult as many of the problems can be solved without algebra, even if they do take longer. The textbook, Maths for all (Maths Education Project 2000a) offers one solution to this dilemma. It has a variety of problems to solve in chapter 1. Later, in chapter 6, it revisits the problems. "In this section, we look at some of the problems again and try to rewrite them in symbol language. This is helpful because the symbol language shows the relationship between quantities in a shorter, 'picture' form. This makes problem solving easier." (Maths Education Project 2000a, p.95) In another Grade 8 mathematics workbook (Hogescholen, 1999, p. 30) the writer goes further: "... you are going to learn more about formula. To start with we are going to forget about the story behind the formula....Just like normal sentences, formulae have grammatical rules." The book then goes on to teach the rules of algebra.

Another way that is suggested to teach skills is that of problem posing. This involves putting forward questions which clarify and focus the thinking about the meaning of the problem and its possible representations. Problem posing is a good way of dealing with problems as well as teaching and consolidating skills. It helps with the understanding of mathematics as well as providing exercises for practice. Walter (1988) gives some practical examples for the classroom. She takes an equation e.g.  $4x - 3 = 24 - 2x$  and asks the students to pose questions. Some could be “What does it mean to solve an equation?; What is meant by an equation?; How many solutions does this equation have?; Where did the equation come from? In trying to solve the equation she encourages the students to write down an equation that they *can* solve and then work on it by adding to or subtracting from both sides of the equation to make it look like the given equation. This is one possible approach to learning to solve linear equations. She comments that at first the teachers and students might find the process of problem posing difficult but that they would soon be doing it unconsciously. The advantage of this method is that it encourages the students to think about their work and to avoid having that blank feeling when faced with a problem. Another advantage is that students can sometimes answer their own problems. Sometimes they will need help and at other times even the teacher will not be able answer the problem. This will help the students to see that mathematics is not an ‘all or nothing and all worked out already subject’. It should also model the teacher as learner.

A particular skill for successful problem solving that will need to be learnt is translation skill. The ability to see a problem when it is represented in different ways is important for attaining an integrated view of the facets of the problem. The students will need to be able to translate from words to symbols, from symbols to symbols and from pictures to symbols. The reading ability of students needs to be of an appropriate standard and they need to be able to read and make sense of mathematical scripts in different contexts. All learners solve problems daily, even ones involving numbers, but when given those same problems in a mathematical context, the learners find them difficult.

Problem solving has been high on the agenda of curriculum developers in South Africa, especially after the results of the Third International Mathematics and Science Study (TIMSS) study. I am growing in my realisation of the importance of the topic. Problem solving should be

an integral part of all mathematics instruction. There can be the danger that teachers who profess to teach using this approach use problems as a motivational device before getting down to the “real algebra”. The benefit of variety and interest is real, but if the problem is used as an excuse to get attention, the students will not become confident problem solvers, but rather narrowly skilled manipulators. This approach needs a teacher who is a problem solver. One who has the ability to take a problem, apply it in the context of her class and construct other tasks from it. It is a way of thinking and of being – which is required of and modeled for her students. Dessart (1983) quotes Carpenter in saying "If computation and algebraic manipulation are the primary objectives of instruction, so that instruction in problem solving is deferred until these skills are mastered, then students are likely to be poor problem solvers."

Kieran (1989, p.36) suggests that for arithmetic word problems the child's mental representation of the problem specifies the operation to be carried out, but that for algebra word problems it is not the solving operation, but the problem situation that must be represented mentally. She gives the example:

"In an existing forest, 425 new trees were planted. A few years later, the 217 oldest trees were cut. The forest then contained 1063 trees. How many trees were there before the new trees were planted?"

A typical example of the children's written work was the statement

$1063 + 217 = 1280 - 425 = 855$  The 217 is added and the 425 is subtracted.

This arithmetical statement shows the typical way of writing down the operations in the order in which they were carried out and keeping a running total. To construct an algebra equation for this situation, the child would have to proceed in precisely the opposite way that she goes about solving the problem, namely  $x + 425 - 217 = 1063$ . In this situation the child has to add 425 and subtract 217.

The particular problems that arise from this approach:

- Problem solving can be taught as following the steps of an algorithm. For example Polya's famous method:
  1. Understand the problem.
  2. Develop a plan.

3. Carry out the plan.
4. Reflect on one's work.

The students just climb into the problem with an algorithm without standing back and looking at it first (Dessart, 1983, p. 37). They may then spend unnecessary time on the problem. To combat this Schmalz (1989) suggests that problem solving must be taught as an attitude as well as a strategy.

- The biggest 'mistake' is not being able to read the problem properly. This results in incorrect statements being written down and then followed through with frustration, as the changed problem cannot be solved.
- Sometimes the manipulation skills are so weak that the translation skills are negated. Beginning algebra students often solve the problem first and then attempt to provide an equation.

#### 4.4. Modeling

The modeling approach requires putting students into a situation that allows them to construct meaning for various representations such as graphs, equations, tables and expressions. They would need to work with them flexibly and then use them to interpret physical phenomena and real world situations. The crucial point in this modeling process, is the formulation phase that results in the creation of the model.

The process of modeling according to Swetz (1991, p. 358) consists of:

- identifying the problem, including the conditions and constraints under which it exists;
- interpreting the problem mathematically;
- using the theories and tools of mathematics to obtain a solution to the problem;
- testing and interpreting the solution in the context of the problem;
- refining the solution techniques to obtain a better answer to the problem under consideration.

The definition of algebra that this approach would agree with is that algebra consists of

- Generational activities which involve generalising from arithmetic, generalising from patterns and sequences, generating symbolic expressions and equations which represent quantitative situations, generating expressions of the rules governing numerical relationships.
- Transformational activities which involve: manipulating and simplifying algebraic expressions to include collecting like terms, factorising, working with inverse operations, solving equations and inequalities with an emphasis on the notion of equations as independent 'objects' which could themselves be manipulated, working with the unknown, shifting between different representations of function, including tabular, graphical and symbolic.
- Global, meta-level activities which involve: awareness of mathematical structure, awareness of the constraints within the problem situation, anticipation and working backwards, problem solving, explaining and justifying. (Brown & Coles, 1999, p. 155)

Modeling is rooted in history. Many of the scientific breakthroughs were a result of this process. For example, in Kepler's time it was assumed that the planets had circular orbits. After accurate observations were made over a long period discrepancies were noticed. Kepler then guessed that the orbits were elliptical. He verified this by using the data and later refined this model into his three laws of planetary motion.

This approach helps the student develop a sense of variable. Variables are either arguments or parameters. Variables truly vary. This approach is in vogue at the moment as it is seen to be applying mathematics in the "real world" by using "authentic situations." The question is whether classroom learning can be structured so that the real mathematics is being learned.

Exponents of this approach would argue that it improves the communication of mathematical ideas, improves the students' mathematical thinking skills and makes them more versatile in problem solving. My difficulty is that one needs to know quite a sophisticated level of mathematics to engage in these activities. It took Kepler a decade to determine an explanation for



the apparent retrograde motion of the planet Mars. There is no indication of how algebraic skills are to be acquired. Alongside problem solving activities modeling must be part of the general algebra curriculum. There might be an argument at this stage between the pure mathematicians and the applied mathematicians as to what mathematics needs to be taught at school as “algebra is intrinsically very general and context-free. The reason why algebra works so efficiently as a calculus is that it treats data impartially and is not concerned with the "meaning" inherent in the context in which the data are situated.” (Wheeler, 1996a, p. 324) He also is critical of this approach as an approach to algebra. “The only way the approach can possibly work is by trivialising it. The reasons why this approach is talked about so frequently now are not mathematical or pedagogical but ideological. The modeling approach has the advantage of seeming “down to earth,” overtly linking “abstract” mathematics to experimental stuff in the “real” world; in this sense it’s non-elitist and a-historical too, so it has the potential to free school mathematics from the constraints of the Eurocentric mold.” (ibid, p.321)

The particular problems that arise from this approach:

- A difficulty that students face in this approach is confusion with the use of units.

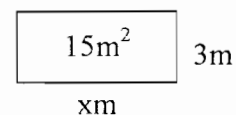
The symbolic designations of magnitude are often letters and this may cause a critical misunderstanding. For example if 2 boys and 5 girls are abbreviated to 2b and 5g in the same way as 5 grams and 4 meters are often shortened to 5g and 7m, then introducing a letter to denote a variable or unknown number clashes with the use of letters to stand for a unit.

A simple example: Area of a rectangle = length x breadth.

In symbols a student might write  $A = lb$ .

In the given diagram  $15m^2 = xm \times 5m$

$$\therefore x = 3m$$



The m is the source of confusion. Is it a variable like the l and the b?

- "Different problems trigger different problem solving strategies, foster different mental images and reveal astonishing different success rates." (Janvier 1996 p. 232) Janvier says that students can sometimes solve a problem in one context but not another even though the problems are conceptually similar.

- A mathematical model is often only an approximation to the real situation. In drawing a graph from raw data, the model could represent the best straight line through the points. This is confusing for the students.
- At the other end of the scale, problems are chosen that can be represented exactly by the model. This might lead the students to believe that all real life problems can be represented exactly.

#### **4.5. Functional Approach**

The functional approach is similar to the modeling approach in that it also requires an ability to deal with real world situations using several representations such as graphs, numeric and symbolic representations. Some mathematics educators consider functions to be the core of algebra. The concept of variables is emphasised in the functional approach. Letters are seen as variables because they can take on a range of values and not unknowns where the letter has a fixed value. The functional approach views functions from the perspective of the relationship between the x-values and the corresponding function values. For example, the expression  $x + 3$  can be viewed as a function, that is a mapping that translates every number  $x$  into another. As such,  $x$  is interpreted as a variable, because it can take on a range of values. In the functional approach, there are usually a greater variety of functional situations (linear, quadratic, and exponential) than one tends to find in standard algebra curricula. The discussions in the classrooms are mathematically rich as the students discuss the relationships between variables and not symbolic routines.

This approach would take a problem and ask the learners to explore it by taking different input values and comparing the output values. This gives the learner a feel for the situation. The learners could then draw up a table of values, draw a graph of the situation, write a formula or describe the situation in words. This approach can also be used to teach algebraic rules by comparing outputs. Equivalent expressions are those which have the same output values for any input value. This table from Mathematics at work (Human et al., 2000a, p.165) could be used to

teach removing of brackets and simplification. This table has not got whole numbers for the variable.

Complete the table:

x	2,3	4,6	-4,6	8,1	0,046	2,008
17 - (3x - 5)						
17 - 3x - 5						
17 - 3x + 5						
17 - (3x + 5)						
12 - 3x						
22 - 3x						
3(4 - x)						

Figure 20: Table to teach removing of brackets

This next table encourages the students to be aware of the different meanings of the algebraic expressions and of the algebraic structure of the expressions, by looking at the output values. (Human et al., 2000a, p.96)

Complete the table:

x	1	2	3	4	5
x <sup>2</sup> + 3					
(x + 3) <sup>2</sup>					
x + 3 <sup>2</sup>					

Figure 21: Table to help students to see the meaning of the expression

The function approach lends itself to use computers. In the book Approaches to algebra: Perspectives for Research and Teaching (Bednarz, Kieran & Lee, 1996), Kathleen Heid (1996) and Kieran, Boileau and Garançon (1996) describe computer programs used to introduce algebra. I include them here as they give some insight into learning algebra using the functions approach.

Kathleen Heid (1996) describes a beginning algebra curriculum that introduces the concepts of functions, variables, equations, inequalities, systems and equivalence in the context of mathematical modeling. This curriculum is called *Computer-Intensive Algebra*. “In this approach to algebra, understanding the concept of function requires being able to solve problems using multiple representations and to reason within and among graphical, numerical and symbolic representations of function. It requires understanding of the properties of basic families of functions and how those properties can be recognised in the different representations.” (Heid 1996, p. 239) This curriculum is written with the assumption that students will have constant classroom access to computing tools, both the hardware and the software. It also depends on a classroom organisation that uses collaborative pairs that work together on the computer. This requirement is based on the belief that conceptual understanding grows more quickly when students are communicating orally with each other about mathematics.

Heid makes the interesting comment that students are not expected to master each concept and procedure when they first encounter it. This view is based on the idea that concepts are learned through a variety of examples which are worked on for a longer period of time and that knowledge is constructed through interactions with the environment. In comparing the performance of students who had been on a *Computer-Intensive Algebra* programme with those who had followed a traditional course, the former were more flexible in their approach to problem solving.

Kieren, Boileau and Garançon (1996) record the use of a software program called CARAPACE which is a functional problem-solving environment that includes three representations: algorithmic, tabular and graphic. Their students were following a traditional algebra course. The researchers worked with different groups of students for an hour or two per week for about three months at a time. In the functional approach that they describe letters are viewed as variables and the representations used for expressing functional relations are presented in an operational, process-orientated way. The context is one of problem solving. In a typical situation, problem solving involves setting up an equation with an unknown. In this functional approach to problem solving, a more general functional relation among the problem givens is set up and the input-

output letter names are viewed as variables. Once the students have been given a problem they try a few numbers to make sense of the problem. When they believe they have figured out the sense of the problem and how to compute the functional values of the situation, they translate their numerical computations into a general formulation. Therefore there is a bit of generalisation in this approach too. The researchers discovered that the students found it easy to write a functional representation of a word problem.

An example of a problem:

Karen works part-time after school selling magazine subscriptions. She receives \$20 as a base salary per week, plus \$4 for each subscription she sells. How much can she earn in a week? How many subscriptions must she sell to earn at least \$50?

Program #1	
<div>Request values for</div>	
Number of subscriptions	
<div>Carry these calculations out:</div>	
Number of subscriptions x 4 <u>gives</u> bonus	
20 + bonus <u>gives</u> salary	
<div>Show values of:</div>	
Salary	

Figure 22: Editing window after program entry

Enter value for "number of subscriptions"

GO

STOP

INPUTS	OUTPUTS	
Number of subscriptions	Salary	
1	24	
2	28	
3	32	
4	36	
5	40	
6	44	
7	48	
8	52	
9	56	
10	60	

Figure 23. Table of values window

Kieren, Boileau and Garançon end their article by asking whether this approach is algebra. It is problem solving. If one uses the definition that algebra is about using literal notation which is accompanied by the rules for transformation on this notation then this approach is not algebra. But it does share with algebra some modes of thinking. For example seeing the general in the particular and thinking in terms of forward operations of solving equations (as opposed to inverse operations) when setting up a problem representation. In algebra one would use the formula  $s = 20 + 4x$ , whereas the computer has to use variables which are words, such as "number of subscriptions", "bonus" and "salary". The authors acknowledge that this approach is an incomplete introduction to algebra.

The particular problems that arise from this approach:

- The *Computer-Intensive Algebra* course does not include the study of pen and paper symbolic manipulation, equations solving or production of equivalent forms. This needs to be taught separately. Kieren, Boileau and Garançon (1996) found the same problem with the CARAPACE programme. They also found that there was no motivation on the part of the students to simplify any expressions as the computer would still carry out the computations quickly.
- The computer tool CARAPACE could usurp the potential of algebra as a problem solving tool. Kieren, Boileau and Garançon (1996, p.267) comment that "...not all technology-supported roads that are intended to be algebraic lead to developing meaning for traditional algebraic representations and transformations. Some of these roads may lead to the creation of alternate problem-solving tools, which may not be unfortunate – especially if one is of the opinion that not all students need a traditional algebra course during their high school careers". This might be an area to consider for the mathematical literacy course intended for Grade 10.
- Students could not arrive at exact solutions to a problem because they lacked the algebraic skills. They were able to produce solutions that were close enough for all practical purposes.
- This approach could be taught without a computer, but it would then need far more time. All the numerical tasks and the graphs would need to be done by hand.
- If the arithmetic skills of the learners are weak, they might not see a pattern in the tables.

#### **Interlude for reflection.**

The literature states that algebra students should be able to make generalisations, solve problems, model situations and work with functions. My approach was predominantly the generalisation approach. The students that I taught had a very one-sided introduction to algebra as I spent most of the time on a narrow slice of this approach, namely, generalised arithmetic. There were only four lessons with functional activities, but as I did not foreground functions, the generalising

might be all that they were aware of. No space was given to the problem solving approach nor was there any opportunity to use algebra to model situations. My own approach to the teaching of algebra largely followed what is called in the literature the traditional approach. This is not an entirely different approach to the four approaches presented above, but rather a particular form of the generalisation approach. There are many references in the literature to this way of teaching, as it seems to be an approach that has been used for a long time.

#### 4.6. Traditional approach

This approach uses one aspect of the generalisation approach. It emphasises the rules for transforming and solving equations. The justification given for this position is that algebra is *generalised* arithmetic. From this perspective the teacher links the arithmetic rules, which the students know to the new algebraic laws. Thus it is hoped that the students are made aware of algebraic structure. Davis (1989) describes the traditional approach as an approach that treats algebra as a collection of highly specific skills and techniques. For example: collecting like terms, removing brackets, multiplying out and solving linear equations. These individual small skills are presented to students as rituals to be practised until they can be executed in the proper orthodox form. Mason (Mason et al. 1985) likens learning algebra like this to being taught how to kick, tap and head a ball without ever knowing the game of soccer. Kieren, et al. (1996) in their article on the functional approach to algebra say that the traditional approach to problem solving is to identify the unknown and then move immediately to setting up the equation. (I call it a “rush to equations”). This prevents the students from becoming aware of all the dynamics of the problem. Mason (1996, p.73) critiques this traditional approach by stating: “At school, algebra has come to mean ‘using symbols to express and manipulate generalities in number contexts’”.

#### Common mistakes

In an article in *Mathematics in School*, Barnard (2002a) lists some of the common algebraic mistakes made by students who follow this approach. He mentions:

$2x - x = 2$	[subtracting the letters only and ignoring the meaning.]
$(xy)^2 = xy^2$	[operating only on one piece of a compound term]
$x(y+z) = xy + z$	[operating only on one piece of a compound term]



$(x + y)^2 = x^2 + y^2$	[squaring only a part of a compound term]
$a^2 + a^3 = a^5$	[adding the numerals only and ignoring the a]
$\frac{x + y}{y} = x + 1$	[inappropriate canceling] (Barnard 2002b)

Barnard suggests that the underlying cause of mistakes is the failure to conceive the objects of manipulation as meaningful ‘things’ in their own right. He comments that "if letters do not have meaning for a student, very little algebra is possible other than fragmentary successful instrumental rote learning of rules." (Barnard 2002a, p.10)

I taught using the traditional way, as it seemed a logical way to me. Mason (1996, p.75) says: "Rushing from words to single letter symbols has marked school algebra instruction for over a hundred years. Formal treatments, which have been very successful for the quick-thinkers and the non-questioners, have dealt with symbols as unknown numbers, and embarked on a series of games with rules for manipulating these. But meaning-seekers and those less able to achieve quick success by recognising simplification patterns spontaneously, have deserted mathematics, maintaining algebra as the principal mathematical watershed for society as a whole." I could identify with this statement as I am a strong pattern maker and therefore I wanted my students to learn to see the patterns quickly. I made some of them go too fast when it came to understanding the rules of algebra. Leila found algebra relatively easy. In a letter that she wrote at the end of the year she said: "You explained it all very clearly (and a long time was spent on the subject), but for those who got it straight away and understood, it was rather boring to sit and wait for the rest."

My assessment of the students’ abilities was done through examinations and tests. My conclusion was that at least half the class could solve equations and manipulate expressions with a degree of success, but there was no indication of understanding or even awareness of any structure. This aspect was not explicitly assessed. Many students needed more time, but I had to watch the clock and chivvy them on as I was constrained by the tight schedules of the timetable and the syllabus.

Algebra is difficult. According to Orton, who follows Piaget's stage theory, the abstract thinking required for algebra is especially hard if not impossible for students who are still at the stage of concrete operations. "Many mathematical ideas require the kind of thinking skills which Piaget has claimed are not available until the onset of the formal operational stage" (Orton, 1992, p. 65). Orton indicates that Piaget describes different levels of intellectual development. The relevant ones for algebraic learning are the concrete operational stage and the formal operational stage. An understanding of algebra requires the students to be at the formal operational stage. Many students only reach that at about 15 years of age and yet we teach them algebra at 13 years. Some students are not ready for abstract ideas. Explanations will only have an impact on the student's algebraic learning if the explanations are dependent on the skills available to students at the concrete operational stage (Orton, 1992). Algebra however requires a student to think and work abstractly, to move beyond the concrete and to think in general terms, leaving behind the particular bit of arithmetic or a given example or context. Algebra requires students to work with the abstract idea behind a special case and to work with the general. But the student needs a sufficient experience with the particular in order to be confident in dealing with generalities.

Having reviewed the four approaches it seems that Orton's warnings based on Piaget's work should caution against a too rapid or early emphasis on purely abstract symbolizing. Notwithstanding this, the phased introduction of patterns can effectively bridge the gap and make it possible for young students to access real algebra learning. However my experience related in chapter 2 indicates that time is a critical constraint. A well-planned teaching programme with links to appropriate support material such as textbooks should help the teacher to make good use of the available time.

## CHAPTER 5 SOUTH AFRICAN MATHEMATICS TEXTBOOKS

In this chapter I analyse some beginning algebra Grade 8 textbooks on the South African market. I characterise each textbook according to the extent to which it develops generalising, encourages problem solving, emphasises functions and models real life. I also consider the definition of algebra given or implied by each book. I have chosen to focus on those textbooks that are currently being used as well as those that are coming new onto the market and that may be selected in the next year or two as new curricula are introduced.

The new textbooks are being introduced at a time when the education system is going through a change towards Outcomes Based Education (OBE). OBE forms the foundation of the new curriculum in South Africa. It strives to enable all students to achieve to their maximum ability. This it does by setting outcomes to be achieved at the end of the process. The outcomes encourage a learner-centered and activity-based approach to education. The long-term goal is expressed in a set of critical outcomes. The critical outcomes envisage learners who are able to:

- identify and solve problems and make decisions using critical and creative thinking;
- work effectively with others as members of a team, group, organisation and community;
- organise and manage themselves and their activities responsibly and effectively;
- collect, analyse, organise and critically evaluate information;
- communicate effectively using visual, symbolic and/or language skills in various modes;
- use science and technology effectively and critically, showing responsibility towards the environment and the health of others; and
- demonstrate an understanding of the world as set of related systems by recognising that problem-solving contexts do not exist in isolation.

The developmental outcomes envisage students who are also able to:

- reflect on and explore a variety of strategies to learn more effectively;
- participate as responsible citizens in the life of local, national and global communities;
- be culturally and aesthetically sensitive across a range of social contexts
- explore education and career opportunities; and
- develop entrepreneurial opportunities.

These critical outcomes set out a target to which each of the eight learning areas is expected to contribute. Mathematics is one of these learning areas. For each learning area there are learning outcomes. Mathematics has five learning outcomes, but the learning outcome that is relevant to this dissertation is Learning Outcome 2: Patterns, Functions and Algebra. The outcome requires that the student will be able to “recognise, describe and represent patterns and relationships, as well as to solve problems using algebraic language and skills” (Revised National Curriculum Statement, 2002). The previous curriculum implemented in the Western Cape in 1997 followed a functions approach. The syllabus included the following processes:

1. Finding the values of the dependent variable. (i.e. finding the function values)
2. Finding the values of the independent variable. (by solving equations)
3. Describing the behaviour of function values. (determining the gradient and where the function was increasing or decreasing)
4. Transforming to a different expression (this involves the manipulation of algebraic expressions.)
5. Finding a function rule (leading to deriving a formula.)

There has been a shift in emphasis from a functional approach towards a problem solving approach in the new curriculum.

Various approaches were discussed in the previous chapter, namely the generalising approach, which included generalised arithmetic; the problem solving approach; the modeling approach and the functional approach. What follows is my analysis of six Grade 8 textbooks that I have chosen. A summary of the content of the various textbooks is in appendix 1.

The question I will ask is whether the text I am considering:

- develops the concept of generalising. Does it only consider algebra as generalised arithmetic?
- encourages problem solving as a natural part of algebra and not just an add on after the rules of algebra have been learned.

- emphasises the 'variableness' of the letters and not consider them as only unknowns. The functions approach also includes different representations of functions: rules, tables graphs and verbal statements.
- takes some real life situations and mathematise them into an appropriate mathematical model. This approach also includes the different representations: rules, tables, graphs and verbal statements.

### 5. 1. Just Mathematics 6 (1991) Fitton S, De Jager C and Blake P . Maskew Miller Longman.

This textbook has a teacher's edition which has a page of explanation at the beginning of each chapter and the answers of the exercises at the back of the book. The writers introduce the students to algebra in chapter 4 with a "think of a number " problem. "Think of a number and write it down. Now add 3, double the result, subtract 4, divide this result by 2, subtract the number you first thought of and write down your answer. They are then encouraged to write the problem in shorthand as it then shows why the answer is always 1. (see figure 12) There are a series of worksheets for the students to work through which require them to generalise using symbolic language. (see figure 17) They are then encouraged to use their generalisation to solve more difficult problems of the same type. This process is represented as showing the students the power of algebra.

Having established the need for algebra, the next chapter deals with algebraic conventions. "Algebra is a language. Every language has its conventions and algebra is no exception." (p60). Algebra can now be seen as generalised arithmetic. The rules of algebra are seen as symbolic statements about the rules of arithmetic.

#### CONVENTION 7

Multiplication is a shorthand form of adding several equal things together.

$8 + 8 + 8 + 8$  can be written as  $4 \times 8$ .

$a + a + a + a$  can be written as  $4 \times a = 4a$

Figure 24 : Example of generalised arithmetic (p. 62)

By the end of this chapter learners should be able to add, subtract, multiply and divide, as well as apply the conventions. The authors encourage teachers to allow the students to 'discover the rules' themselves, and to make their own generalisations. " Pupils should be encouraged to work from first principles. Eventually generalisations or rules will develop in the subconscious." (p. 58b). In Chapter 7 the students learn how to translate from English to algebra and to solve simple equations. These skills are then applied to solving problems. Problem solving is a stated aim of this chapter, but as an introduction to algebra, the acquiring of skills is foregrounded. The problems are mainly mathematical problems. "I have written down three secret numbers. The first number is 10 bigger than the second number and the third number is twice the size of the second number. They add up to 310. Find them." (p. 86). Integers are introduced in Chapter 10 and fractions are revised in chapter 11. The rules and concepts of algebra are revised again in these chapters.

The development of algebra as generalised arithmetic is on the whole rather more implicit than explicit. This textbook has problems to be solved, but these are used as an application to equations. There are no examples of modeling or of functions.

## **5.2. On Track With Mathematics. Grade 8 (2000) Fitton S, De Jager C and Blake P Phumelela Books**

This purports to be an OBE update of Just Mathematics. The examples and the approach are almost the same. The book's cover and the typeface used are different, but the new book has the same content as the 1991 book. The emphasis on self-discovery in the OBE framework now makes the investigations in this book important. One of these worksheets is illustrated below.

### INVESTIGATION 1

1. Look at this sequence of numbers and write it down  
1; 3; 4; 7; 11; 18; 29;...  
When you have found the pattern, fill in the next three terms.  
Now explain how each term of the sequence from the 3<sup>rd</sup> term  
Onwards is found.
2. (a) Add the ten terms and (b) multiply the 7<sup>th</sup> term by 11.  
What do you find?
3. Now make up another sequence of ten terms following the same  
type of pattern starting with 2; 3;... Follow the same  
instructions as in question 2.  
What do you find?
4. Now make up a third sequence of ten terms following the same  
pattern. The choice of the first two terms is open to you. Repeat  
the same instructions as in question 2.  
What did you find?

Try to explain in words or in some other way why this happens.

Figure 25: Investigation of a pattern (p. 59)

### 5.3. Classroom Mathematics Std 6 (1983) Laridon P. Kitto A. Pike M. Strimling L. Burgess A. Jawurek A. Strauss J. Wilson H. Lexicon Publishers Johannesburg

Algebra is introduced with a story that shows the usefulness of algebra as a generalising activity.

The story goes like this:

"As was customary amongst the Arabs in those days, the elderly desert chieftain had given each of his three sons a purse of silver coins, a camel and some provisions, and had sent them into the city to make their fortune. It was now exactly one year later, and the three young men stood before him. Confronting the eldest son, the chieftain was pleasantly surprised when the young man handed him back his 30 silver coins which had been given to him. Soon after arriving in the city the son had sold his camel for 6 silver coins. The money that he then had, had been doubled by careful trading during the year. However, he had found it necessary to spend half his total money on provisions for the journey home as well as a further 3 coins on the purchase of a

camel. The old chieftain slowly counted the coins and carefully traced the steps of his eldest son's transaction in the sand. Why do you think he was pleasantly surprised? Now do this calculation yourself and work out the profit that the eldest son had made."

The story continues with the second and third son making exactly the same transactions and each time returning their original amounts to the father. The second son had been given 20 coins and the third son had been given 10 coins. Each time the students are encouraged to work out the profit and comment on the father's reaction. The story ends "If you have been careful in your calculations you will see that although each son had started with a different amounts of money and had performed exactly the same set of transactions, they had each ended up with a profit of three silver coins. It is not surprising that the old chieftain found it tedious doing these transactions. Do you agree that it is time consuming to perform so many calculations when they all have the same answer?"

There follows a description on how to do the calculation where the amount of money is represented by an "n".

Algebra is introduced as generalised arithmetic. The rules of algebra are seen as a symbolic restatement of the rules of arithmetic. It is logical as it is seen to be building onto knowledge that the students have. The book introduces variables and operations in the following way (p43):

The sum of five and four	$5 + 4$
The difference between five and four	$5 - 4$
The product of five and four	$5 \times 4$
The quotient of five and four	$\frac{5}{4}$
The sum of five and a number	$5 + x$
The difference between five and a number	$5 - x$
The product of five and a number	$5 \times x$
The quotient when five is divided by a number	$\frac{5}{x}$

Figure 26: The introduction of the variable



The word variable is introduced as the unknown with examples to illustrate its meaning and use. The students are then taught the use of variables and the four basic operations: sum, difference, product and quotient. The chapter on equations is introduced by the statement: “An equation is a statement that two quantities are equal. Determining the numerical value(s) of the unknown quantity (represented by a letter) is called solving the equation.” (p. 186). Then follows a description on how to solve equations, with examples to practise. This equation solving skill is then applied to solving simple problems. The writers of this textbook would go along with the definition of algebra given in the Chambers Twentieth Century Dictionary. “Algebra is a method of calculating with symbols - by means of letters employed to represent quantities, and signs to represent their relations, thus forming a kind of generalised arithmetic.” (Macdonald, 1972)

As in Just Mathematics, the development of algebra as generalised arithmetic is on the whole rather more implicit than explicit. Classroom Mathematics follows more the traditional approach where the students can read the explanation and practise the many examples until they are proficient. A key feature of this textbook is that it has a large number of exercises for the students to practise their algebraic skills. There are problems to be solved, but these are used as an application to equations. There are no examples of modeling or of functions.

#### **5.4. Outcomes-based Classroom Mathematics Grade 8 (2000) Laridon P.**

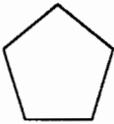
**Barnes H. Jawurek A. Kitto A. Pike M. Sigabi M. Tebeila S. van Noort D. Wilson H. Heinemann, Sandton.**

In this revised version of the 1983 book there is even a greater emphasis on algebra as generalised arithmetic. Much more emphasis is placed on self-discovery using arithmetic as a justification of what is done in algebra (see figure 13). In chapter 4 algebra is introduced using an analysis of patterns in order to find a formula. The word variable is defined as “a letter of the alphabet that stands for a number in a mathematical expression.” (p. 62). The algebraic conventions are then introduced with a strong emphasis on the *generalised* arithmetic. In this book integers are dealt with first in chapter 3. Chapter 7 introduces the students to exponents. Equations are dealt with in chapter 14. An equation is now “a mathematical statement that uses the equal sign (=).” (p. 255). Students are introduced to the balancing method and the

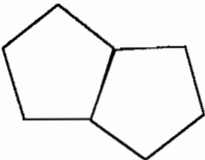
transposition method as algebraic methods of solving equations. This equation solving skill is then applied to solving simple mathematical problems.

Functions are introduced in chapter 6. Functions show a relationship between variables, and variables can now be dependent or independent. Students should be able to see a pattern, complete a table, write the formula and then plot the graph from the table.

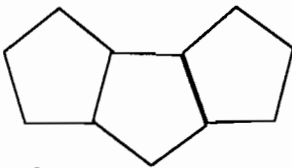
2. Use this pattern of pentagons to answer the question below.



1 pentagon



2 pentagons



3 pentagons

a) Copy and complete the following table:

No of pentagons (x)	1	2	3	4	5	6	7	8	9	10
No of matches (y)										

b) Write down a general formula for the relationship between the number of pentagons (x) and the number of matches (y).  
c) Draw in the x- and y-axes on the graph grid (not included here) and label them. Allow for y-values up to 41.  
d) Plot the ordered pairs in question2 a) onto the graph grid to represent the relationship as a graph.

Figure 27: Example of working with functions (p. 104)

Generalising is the dominant approach of this book, but it does include modeling and functions in the chapter on graphs. Problem solving is done only in the context of equations. In comparison with the earlier book, this book has fewer examples for the students to work through.

**5.5. Mathematics At Work Grade 8 (2000) Human P. Olivier A. le Roux A.  
Bennie K. Sasman M. Liebenberg R. Sethole G. Nasou: Cape Town**

This book was written for the implementation of OBE, which was introduced into the secondary schools in 2001. There is a teacher’s guide that accompanies the Learner’s book. The writers have approached algebra from a functional position. Functions are used to describe relationships between variables. The writers have also included modeling and some generalising. The book starts by introducing the students to a wide variety of relationships using flow diagrams and tables. No letters are introduced. Students are encouraged to write formulae in words.

1. The cost of hiring a bus to travel from Johannesburg to Pietersburg and back is R750.  
(a) If 30 people go on the trip, what must each passenger pay if they are to share the cost equally?  
(b) If 15 people go on the trip, what must each passenger pay if they are to share the cost equally?  
(c) Complete the table:

Number of passengers	5	10	15	20	30	40	50	n
Cost for one passenger								

(d) Write a formula (your method in words, in symbols or a flow diagram) to calculate the cost for one passenger for any number of passengers travelling on the bus.

Figure 28: An example of a relationship. (p. 9)

Chapter 2 introduces them to the concept of a function, which is described as a special relationship. In chapter 7 the students are introduced to algebraic language. The conventions of algebra are taught through the input-output table (see figures 20 and 21). By completing them it is hoped that the students get an understanding of the meaning of the algebraic expression. Chapter 12 extends their algebraic knowledge to equivalent expressions. These activities should motivate them to simplify expressions. Equations are dealt with in chapter 12 and chapter 16. Chapter 11 gives the students opportunities to explore functions with graphs. At the end of this chapter they should be able to translate between words, tables formulae and graphs.

When it comes to dealing with the algebraic symbols, the authors state that their approach is to develop concepts that move from an operational conception of algebra towards a structural conception (Human et al., 2000b p. 60). This reflects the recommended approach of Kieran who distinguishes these same categories. The “procedural refers to arithmetic operation carried out on numbers to yield numbers.... they yield a numerical result.... Structure refers to a different set of operations that are carried out, not on numbers, but on algebraic expressions.... The operations that are carried out are not computational. Furthermore, the results are yet algebraic expressions.” (Kieran, 1992, p. 392) This book seems to have a broad and sound approach to the teaching of algebra in that it exposes the students to these essential aspects of algebra. They will know about modeling, generalising and problem solving all from the point of view of understanding and being able to work with functions.

The book aims to develop an advanced level of algebra. The students are introduced to

factorising as well as being taught to solve equations like  $\frac{1}{2}(7x - 4) = \frac{2(x + 5)}{3}$  which is usually only learned in grade 9.

#### **5.6. Maths for all Grade 8 (2000a) Maths Education Project. Macmillan.**

This book has an accompanying teacher’s resource book. The first chapter consists of a variety of problems. For example:

Two boxes of apples have each an unknown number of apples in them. The total number of apples is less than 20. You must use the clues given below to work out this unknown number.

- Clue 1: If one apple is taken from box X and placed into box Y, then box Y will have double the number of apples that box X will have.
- Clue 2: If, instead, 1 apple is taken from box Y and placed into box X, the two boxes will have the same number of apples.

How many apples are in each box? Show all your work and explain your method or strategy.

Figure 29: Example of a problem (p. 12)

The students are encouraged to solve these using any means that they choose in consultation with their fellow students. The students are then encouraged to discuss their different strategies.

Chapter 6 has the title "From numbers to letters". Letters are introduced to stand in for unknown amounts. The letters are used descriptively as in formulae - not the ubiquitous  $x$ ,  $y$  or  $z$ . The problems are used as a motivation for using algebra. The problems of chapter 1 are revisited, but now the students are encouraged to use letters.

Two boxes of apples are shown. Both have an unknown number of apples. We will use the symbol  $X$  to stand for the number of apples in the one box, and  $Y$  to stand for the number of apples in the other box.

...  
Symbol sentences allow you to rewrite problems in a shorter form that shows the relationship between the quantities. They also allow you to calculate unknown quantities like  $X$  and  $Y$ .

Figure 30: The introduction of letters. (p. 95 and 96)

The book includes very little of the traditional algebra manipulations in Grade 8. The algebraic structures are emphasised and there is no "rush to symbols". The authors' view of algebra is that it is "letter arithmetic", a language that allows one to generalise, to show structure and to communicate in a precise and unambiguous way using mathematics. The ultimate aim is to see the structures of problems, to write general statements in terms of symbols and then to solve the problems. The book uses tables as one of the strategies to see the patterns as a way to solve problems. Chapter 11 has the title Graphs, the picture relationships. The emphasis is on the relationships and interpretation of the graphs rather than on equations and formulae.

This book has an interesting variety of problems to solve. The teacher can select some of them or do them all. There is flexibility in the suggestions in the teachers' handbook. For example they say that the teacher could go straight to the algebra chapter after solving the problem in chapter 1. Algebra is introduced very slowly in this series of textbooks. In speaking to one of the writing team, I was informed that even in grade 9 their series is light on algebra. It was a conscious move to only handle formal algebra in Grade 10.

### **Interlude for reflection.**

I used the textbook Just Mathematics 6. I followed the textbook closely in the order of presentation and the students worked through all the exercises in the book. I supplemented the exercises when I wanted the students to practise more exercises of the same type that were in Just Mathematics 6. The textbook follows a traditional approach of learning the algebraic manipulations in a decontextualised environment. In terms of the four approaches, it follows the generalised arithmetic approach. I used that approach most of the time. However when I introduced algebra to my class I used some exercises from Mathematics At Work. These exercises emphasized a functions approach. The class only worked on these exercises for four lessons and that was the extent to which they were introduced to functions. After a break from algebra, the students learned how to manipulate equations in order to find the value of “ $x$ ”. Just Mathematics 6 does have a few of what I have called mathematical problems, but I did not present these to the class at all. It was only when I had to analyse the textbooks carefully that I read all the comments to teachers in the teacher books. I admit to having read the intentions of the textbook writers of Just Mathematics 6 for the first time. Now when I read them again I can see how I did not use the textbook as it was intended. Instead of leading them into concepts, I gave them the rule and told them to practice it, rather than letting the students discover the rule themselves, as the textbook says they should.

Working with Just Mathematics 6 in the way that I did, gave the early learner of algebra a narrow view of the subject. A student of algebra should end up being able to use algebra to solve problems, to model situations, to handle functions, and to make generalisations. The book, Maths for All, which best exemplifies the problem solving approach has only a small amount of formal algebraic content. The one that does deal with functions, Mathematics at Work, has relatively little variety in the problem solving beyond its emphasis on dealing with functions. As such its approach to problem solving appears to be narrow. If one chooses Just Mathematics, On Track with Mathematics or Classroom Mathematics, the students could lose out on functional understanding. In my analysis of the textbooks, I commented that Mathematics At Work seemed to be the best in terms of including at least something of each of the four approaches. I would however find this textbook difficult to use all the time. The activities for algebra are too similar to one another. Students are repeatedly required to complete tables and identify patterns. In a

heterogeneous class some students would work fast and get them all right while some other students would struggle with the arithmetic and get very little right. I found it difficult to consolidate knowledge and skills when the students were working at such different rates. Heid (1996, p. 249) identifies with this problem when she talks about teachers “dealing with the difficulty of drawing closure in a class whose members are moving at widely varying rates”. For me this issue remains unresolved.

The table below gives a weighted analysis of the reviewed textbooks according to the four approaches. I gave *Just Mathematics 6* four stars for its strong emphasis on generalising, and then all the other ratings were given in relation to that. The first four books on the list are all of the same traditional genre emphasising generalising. *Mathematics at Work* deals with each of the other approaches via functions. In the book *Maths for All*, all four approaches are represented, especially problem solving, but the book makes very little use of algebraic symbols.

Textbook	Generalising	Problem solving	Functions	Modeling
Just Mathematics 6	* * * *	*	-	-
On track with Mathematics	* * * *	*	-	-
Classroom Mathematics Std 6	* * * *	*	-	-
Outcomes-based Classroom Mathematics	* * * *	*	-	-
Mathematics at Work	*	* *	* * * *	* *
Maths for All	*	* * * *	*	* *

Figure 31: Characterisation of the textbooks in terms of the four approaches.

There is a wide range of textbooks in the South African market. It is a difficult for a mathematics teacher to make a choice which provides a balanced approach to algebra and which is appropriate to the background and levels of her students. What she prefers will depend on her own view of algebra and her interpretation of the curriculum.

## CHAPTER 6 CONCLUSION

The recommendation from the previous chapter was for the use of a varied approach when introducing algebra. This recommendation rests on a view of the nature of algebra, which sees the importance of its varied facets for a beginning learner. From the broad definition of algebra I came to see its wide scope. In looking back I see that I was limiting my students' view of algebra by presenting it within the confines of a narrowly defined generalising activity.

All Grade 8 students are obliged to learn algebra. In Grade 9 they have to pass mathematics in order to progress. It would be so nice to be able to have a neat method of teaching it that always works. In the second chapter I recorded the experience of my Grade 8 students learning algebra over two periods totaling eleven weeks. But as the literature says, algebra is hard. Even the experts disagree as to the best method to go about teaching it. There can be a number of different ways of starting algebra while placing the emphasis on any one of the four particular approaches. All four approaches discussed in this dissertation are needed in any algebra programme.

There are those that suggest that we should introduce algebra through generalising activities. The first stage of this is 'seeing' or mentally grasping a relationship or pattern. This is followed by a second stage of 'saying' it, where the student makes an attempt to articulate the insight in words. Skemp (1971) puts this another way when he talks about intuitive and reflective intelligence. Intuitive intelligence is required for the action of working through an example, seeing a pattern or solving a problem. The reflection is shown in the ability to explain how it was done. Once the student is able to reflect on her work she can then start making generalisations. "This process of mathematical generalisation is a sophisticated and powerful activity. Sophisticated, because it involves reflecting on the form of the method, while temporarily ignoring its content" (Skemp 1971, p. 61). When algebra is seen as generalised arithmetic, then the symbols would represent a way of reflecting on the structure of the arithmetic operations.

Problem solving is a priority in the South Africa's Curriculum 2005 and for mathematics in particular. According to the Third International Mathematics and Science Study (TIMSS) study, South African children are not good at solving contextual problems. A problem solving approach



should be rich in providing a variety of contexts for thinking mathematically. Therefore it needs to be taught by a teacher who is a problem solver or one who has the willingness to learn to solve problems. A practical approach is problem posing or asking questions. This encourages students to reflect on the particular problem at hand from different points of view by asking questions. This deepens understanding of the problem itself and so may lead towards a comprehensive solution.

The modeling approach encourages the use of a variety of representations of problem situations. Students are helped to become comfortable with moving from one representation to another. This approach is seen to be applying mathematics to the "real world" by using "authentic" situations. Often intuition is needed to identify the situation and to represent it mathematically in an appropriate model (e.g. a table or a graph). The student should then test the results, which are given by the model and interpret the solution by referring back to the real world situation. The model may then be refined in order to obtain the best solution.

The concept of variable is emphasised in the functional approach. A function denotes a relationship between the variables. A student could get a feel for the problem by looking at how one of the variables changes with respect to another variable. This approach is also said to be rich in providing the context for thinking mathematically. The use of computers allows the students to investigate functions without being hampered by many calculations. Verbally stated problems are translated into algebraic expressions, which are subsequently graphed.

Textbook writers use the different approaches in introducing algebra in currently available Grade 8 textbooks. There are a wide variety of textbooks on the market at the moment as there are changes in the curriculum. The algebra content for Grade 8 has not changed substantially, but emphases have changed somewhat. Broadly speaking a functional approach is still evident but problem solving in various guises has now joined it.

My own teaching has followed what is regarded as a traditional approach. I used most of the activities in the *Just Mathematics 6* textbook, which has an emphasis on manipulation. The mistakes made by my students were typical of those made by beginning learners of algebra. This

study covered their initial learning of algebra, and I hope that as they encounter algebra in Grade 9 again they will become increasingly aware of the structures of algebra. Kieran (1992) comments that students resort to memorising rules and procedures to cover their lack of understanding and that they eventually come to believe that this activity represents the essence of algebra. She says that research suggests that most students do not acquire any real sense of the structural aspects of algebra. There needs to be a lot of practice on procedures before a concept of the algebraic structures becomes clear.

### **The way ahead**

I am feeling embarrassed about my teaching as reported in this dissertation. I taught algebra with little understanding of the essence of the subject. I could apply the rules and enjoyed the manipulations. But I don't think I had much understanding of the way to teach it so that the students would have a good grasp of the subject. I can understand now why I wanted to investigate how my students learned algebra. I did not have the answers. Through my reading I have gained insights into the subject, but it will need a lot more effort to work through a methodology to teach. I have goals for the cognitive development, for assessment and for changing the classroom dynamics. These are great ideals, but change takes time. When the time pressure is on, what will prevent me from slipping back into my old ways? As far as the content is concerned I have already made some small changes in my teaching in 2003. I introduced negative numbers on their own, without the confusion of the letters. In my introduction to algebra I still used the patterning, but I foregrounded the functional approach, focusing on the input and output values. Then I looked at generalising and finding the rule for the pattern. This is the difference which has come in as a result of my research.

Building on this I wish to:

1. design the content of an introductory algebra course that covers generalising, functions and problem solving. I would consider including some elementary modeling while doing the problem solving.

2. investigate other forms of assessment that test more than whether the student can manipulate algebraic expressions. I would have to develop assessment tools which highlight the understanding of functions and generalizing, and the skills of problem solving.
3. find out how to teach algebra to a mixed ability class, to cater for the students who need more time as well as those who pick up the patterns easily.

As Kieran has said, teachers who are reconsidering their teaching have to hunt down research relevant to the classroom. “In-service courses on algebra teaching and collaboration with researchers, though effective in raising teachers’ awareness levels of other approaches worth trying in their classrooms, are even more difficult to find” (Kieran, 1992, p. 413). And then I ask myself the question: “What about other topics in the mathematics syllabus?” What do I know about introducing negative numbers? How do I teach geometry to someone who cannot see the angles? University level courses on understanding school mathematics are rare. I would have to do similar reading and research in these other fields.

The Grade 8 students of 2002 and I embarked on a journey. Our experiences have changed us. We parted company at the end of that year. In 2003 I only have three of my original students in my Grade 9 class. We will travel further together. I also have another two back in the Grade 8 class. A further possible area of research would be to track the developing grasp of algebra by students of the above class during the rest of their school career.

One incident helps me to trust that at least attitude can change over time. I had just finished revising some algebra with the Grade 8's in the previous lesson. The work was still on the board when the Grade 11's came in to the classroom. The following examples were on the board:

$\begin{array}{r} \text{Subtract: } 5a \\ -2a \\ \hline 7a \end{array}$	$\begin{array}{r} \text{Subtract: } 3x^2 - 2x - 6 \\ 4x^2 - 5x + 3 \\ \hline -x^2 + 3x - 9 \end{array}$	$\begin{aligned} &3x^2 - 2x - 6 - (4x^2 - 5x + 3) \\ &= 3x^2 - 2x - 6 - 4x^2 + 5x - 3 \\ &= -x^2 + 3x - 9 \end{aligned}$
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Figure 32: Work on the chalkboard.

The conversation went like this:

One Grade 11: "I wish we could have things like this in our maths exam."

Me: "Is it easy?"

Chorus of Grade 11's: "Yes!"

Me: "Did you find it easy in Grade 8?"

Grade 11's: "No!"

Unfortunately I did not carry on the conversation, but that night when I was reflecting on the day, I wondered at which stage did it become part of their own knowledge, or whether as one mathematician said: "You never really understand mathematics, you just get used to it."

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## CHAPTER 7 REFERENCE LIST

- Barnard, T. (2002a). Hurdles and Strategies in the Teaching of Algebra (part I). *Mathematics in School*, Vol.31, No 1. January 2002.
- Barnard, T. (2002b). Hurdles and Strategies in the Teaching of Algebra (part II). *Mathematics in School*, Vol.31, No 4. September 2002.
- Baumgart J. K (1969). The History of Algebra. In *Historical Topics for the Mathematics Classroom. Thirty-first yearbook of the National Council of Teachers of Mathematics*. (pp. 233 – 260). NCTM: Washington D.C.
- Bednarz N., Kieran C. & Lee L. (Eds.). (1996). *Approaches to Algebra: Perspectives for Research and Teaching*. Kluwer Academic Publishers: Dordrecht.
- Boyer, C. (1968). *A History of Mathematics*. John Wiley: New York.
- Brown, L. & Coles, A. (1999). Needing to use Algebra. In O.Zaslavsky (Ed.) *Proceedings of the 23<sup>rd</sup> Conference of the International Group for the Psychology of Mathematics education*, Vol. 2, (pp. 153 – 160). Haifa, Israel.
- Cummins, K. (1969). Equations and the Way They Were Written. In *Historical Topics for the Mathematics Classroom. Thirty-first yearbook of the National Council of Teachers of Mathematics*. (pp. 260 – 263). NCTM: Washington D.C.
- Davis, R.B. (1989). Three Ways of Improving Cognitive Studies in Algebra. In S.Wagner, & C. Kieran, (Eds.), *Research Issues in the Learning and Teaching of Algebra*. NCTM. L.Earlbaum Associates: Hillside, N.J.
- Dedron, P. & Itard, J. (1959). *Mathematics and Mathematicians 2*. Transworld Publishers: London.
- Dessart, D.J. (1983). Algebra. In D. Dessart, & M. Suydam, (Eds.), *Classroom ideas from research on secondary school mathematics*. (pp. 29 – 38)NCTM: Reshton, Va.

Fitton S., De Jager C. and Blake P. (1991). *Just Mathematics 6*. Maskew Miller Longman: Cape Town

Fitton S., De Jager C. & Blake P. (2000). *On Track With Mathematics. Grade 8*. Phumelela Books: Cape Town

Heid, M.K. (1996). A Technology-Intensive Functional Approach to the Emergence of Algebraic Thinking. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching*, (pp. 239 – 255). Kluwer Academic Publishers: Dordrecht.

Hewitt D. (1998). Approaching Arithmetic Algebraically. *Mathematics Teaching*, 163, pp. 19-29.

Hogben, L. (1937). *Mathematics for the Million*. 2<sup>nd</sup> ed. George Allen & Unwin: Woking.

Hogescholen, F. (1999). *Grade 8 Algebra*. IMSTUS: University of Stellenbosch

Human P., Olivier, A., le Roux, A., Bennie, K., Sasman, M., Liebenberg, R., and G. Sethole (2000a). *Mathematics At Work Grade 8 Learner's Book*. Nasou: Cape Town.

Human P., Olivier, A., le Roux, A., Bennie, K., Sasman, M., Liebenberg, R., and G. Sethole (2000b). *Mathematics At Work Grade 8 Teacher's Book*. Nasou: Cape Town.

Janvier (1996). Modeling and the Initiation into Algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching*, (pp. 225 – 236). Kluwer Academic Publishers: Dordrecht.

Kieran, C. (1989). The Early Learning Of Algebra: A Structural Perspective. In S. Wagner & C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*, (pp. 33 – 56). NCTM: Hillside, N. J.

Kieran, C. (1992). The Learning and Teaching of School Algebra. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, (pp. 390 – 419). NCTM. Macmillan: New York, N. Y.

Kieran, C. (1994). A Functional Approach To the Introduction Of Algebra – Some Pros And Cons. In J. Pedro & J. Matos. (Eds.) *Proceedings from the 18<sup>th</sup> Conference of Psychology of Mathematics Education*, Vol. 1, (pp.157 – 175). Lisbon, Portugal.

Kieran, C., Boileau, A., & Garançon, M.. (1996). Introducing Algebra By Means of a Technology-Supported, Functional Approach in Approaches to Algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching*, (pp. 257 – 293). Kluwer Academic Publishers: Dordrecht.

Laridon P., Barnes H., Jawurek A., Kitto A., Pike M., Sigabi M., Tebeila S., van Noort D. and H.Wilson (2000a) *Outcomes-based Classroom Mathematics Grade 8 Learners Book*. Heinemann: Sandton.

Laridon P., Barnes, H., Jawurek, A., Kitto,, A., Pike M., Sigabi, M., Tebeila, S., van Noort, D. and Wilson, H. (2000b) *Outcomes-based Classroom Mathematics Grade 8 Teachers Book*. Heinemann: Sandton.

Laridon P., Kitto A., Pike M., Strimling L., Burgess A., Jawurek A., Strauss J., and H. Wilson (1983). *Classroom Mathematics Std 6*. Lexicon Publishers: Johannesburg.

Linchevski, L. (2002). Lecture at University of Cape Town. 31 January 2002.

Macdonald, A. M. (ed.) (1972). *Chambers Twentieth Century Dictionary*. W & R Chambers Ltd: Edinburgh.

Mason, J. (1996). Expressing Generality and Roots of Algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching*, (pp. 65- 86). Kluwer Academic Publishers: Dordrecht.

Mason, J., Graham, A., Pimm, D. & Gowar N. (1985). *Routes to / Roots of Algebra*. Centre for Mathematics Education: The Open University, Milton Keynes.

Maths Education Project (2000a). *Maths for all Learners Book Grade 8*. Macmillan: Manzini, Swaziland.

Maths Education Project (2000b) *Maths for all Grade 8 Teachers Book*. Macmillan: Manzini, Swaziland.

Mengheni M (1994). Form in Algebra: Reflecting, with Peacock, on Upper Secondary School Teaching. *For the Learning of Mathematics* **14**,3. November 1994 (pp. 9 – 14). FLM Publishing Association: Vancouver, British Colombia, Canada.

Orton, A. (1992). *Learning mathematics: Issues, theory and classroom practice 2<sup>nd</sup> ed.* Cassell: London.

Revised National Curriculum Statement: Grade R – 9 (Schools): Mathematics.(2002). Pretoria: Department of Education.

Schmalz, R. (1989). Problem Solving – an Attitude as Well as a Strategy. *The Mathematics Teacher*, Vol. 82, no. 9, (pp. 685 – 687) December 1989.

Skemp, R.R. (1971). *The Psychology Of Learning Mathematics*. Penguin Books: Harmondsworth.

Swetz, F. (1991). Incorporating Mathematical Modeling into the Curriculum. *The Mathematics Teacher*, Vol. 84, no. 5, May 1991.(pp. 358 – 365)

Wagner, S. & Kieran, C. (1989). An Agenda for Research on the Learning and Teaching of Algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*. NCTM.: L.Earlbaum Associates: Hillside, N.J.

Walter, M. (1988). Some Roles of Problem Posing in the Learning of Mathematics. In D. Pimm (Ed.), *Mathematics Teachers And Children*. (pp. 190 – 200). Hodder and Stoughton: London

Wheeler, D. (1996a). Backwards and Forwards: Reflections on Different Approaches to Algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching*. (pp. 317 – 325) Dordrecht: Kluwer Academic Publishers.



Wheeler, D. (1996b). Rough or smooth? The Transition from Arithmetic to Algebra in Problem Solving. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching*. (pp.147 – 149) Dordrecht: Kluwer Academic Publishers.

Wong, M.P.H. (1997). Numbers Verses Letters in Algebraic Manipulation: Which is More Difficult? In E. Pehkonen (Ed.) *Proceedings from the 21<sup>st</sup> Conference of Psychology of Mathematics Education*, Vol.4,( pp. 285 – 289). Lahti, Finland.

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## APPENDIX 1.

### Summaries of the algebraic content of the textbooks

#### 1. Just Mathematics 6 (1991) Fitton S, De Jager C & Blake P . Maskew Miller Longman.

##### Chapter 4 Mathematical patterns.

- Starts with a 'Think of a number' problem. Shows that there is a better way to write a problem using algebra. The authors then present some investigations using patterns where the students are encouraged to write down the rule.

Chapter 5. Algebra. "You still have to **learn** the language. Then you will be able to follow....Algebra is a language. Every language has its conventions and algebra is no exception." (p. 60)

- Conventions of algebra:
  - Do not use a  $\times$  or  $\div$  sign in algebra:  $3 \times p = 3p$ ,  $\frac{4}{a}$  for  $4 \div a$
  - Always write the number in front of the letter:  $3ab$
  - But we leave out the 1 in front:  $1p = p$
  - $k \times k = k^2$ ,  $a \times a \times a = a^3$
  - $3a = 3 \times a$  or  $a + a + a$
  - Write terms in alphabetical order  $cab$  should be  $abc$
  - Multiplication is a shorthand of addition.  $a + a + a + a = 4a$
- The rules they needed to know were:
  - Adding and subtracting e.g.  $3x + 5x = 8x$
  - Multiplying and dividing e.g.  $3x \times 5x = 15x^2$
  - Like and unlike terms: can only add and subtract like terms e.g.  $3a - 2b$  cannot be simplified.

- Terminology terms, coefficient, types of terms, ascending and descending order of powers, order of working.

#### Chapter 6. Formulae

- Writing formulae from diagrams, substitution.

#### Chapter 7. The language of algebra

- Writing verbal expressions into algebraic expressions.
- Practice solving equations (inspection and inverse operations)
- Translate problem sums into equations.

#### Chapter 10. Directed numbers

- Each rule (adding, subtracting, multiplying, dividing) starts with an exercise with numbers and then goes on to exercises with letters.
- Removing brackets is also introduced here. e.g.  $2(s + t) = 2s + 2t$

#### Chapter 11 Fractions

Also starts with numbers and then goes on with letters

#### Chapter 12 More difficult equations

Solves equations by

- trial and error
- balancing
- transposing

The chapter ends with problems leading to simple equations. For example 'Find two consecutive numbers such that 6 times the smaller is 18 more than 5 times the greater.'

## **2. On Track With Mathematics. Grade 8 (2000) Fitton S, De Jager C & Blake P Phumelela Books**

The content is the same as Just Mathematics

### 3. Classroom Mathematics Std 6 (1983) Laridon P et al. Lexicon Publishers

The book tackles integers after introducing algebra

#### Chapter 3 Introduction to algebra.

Introduces Algebra as a generalising activity. As with the above Just Mathematics book, Laridon's book gives a word problem type "think of a number..." The authors then write the instructions in terms of a letter, say  $x$ , and then show why the strategy works. The letter, called a variable, stands for a number. The replacement set is the set of numbers, depending on the context.

Chapter 3 then goes on to "study the use of variables and the four basic operations."

Exercises:

- Write in algebraic notation
- Convention of  $\times$  sign
- Terminology: - coefficient, number of terms,
- Writing an algebraic expression from a phrase
- Adding and subtracting
- Substitution
- Developing formulae from tables.
- Like and unlike terms

#### Chapter 5 Powers and exponents

Exercises:

- Multiplying and adding e.g.  $ab + ab + ab$  and  $ab \times ab \times ab$
- Multiplying when the bases are the same
- Raising to a power
- Multiplying a polynomial by a monomial
- Division
- Division of a polynomial by a monomial

#### Chapter 6 The integers

Integers are introduced in this chapter and algebra is revised.

#### Chapter 7 Equations

- Solve equations by inspection and inverse operations
- Equivalent equations - using equivalent expressions to develop the rules for solving equations: do the same to both sides of the equations.
- Problems: write in algebraic form and solve the equation.

#### Chapter 9 The rational numbers

Fractions are the subject of this chapter. Algebra is revised again.

### 4. **Outcomes-based Classroom Mathematics Grade 8 (2000) Laridon et al. Heinemann**

This later version of the Laridon text now covers integers before handling algebra.

#### Chapter 4 Generalising number patterns

The stated approach of the chapter is “Algebraic formulas are observed from observable social phenomena. This develops the critical idea by basing the materials/worksheets on real life activities.” (2000 b p. 32)

- Generalising from patterns; predicting patterns
- Algebraic expressions
- Discovery of rules: (Rules are 'discovered' by looking at what happens in arithmetic.)  
adding and subtracting like terms, commutative and distributive laws, multiplication and division of expressions (using tables to motivate)

Three or four examples of each type are given to practice

The chapter aims to equip students to deal with the following Assessment Criteria:

- ✱ Uses variables to express generalisations

- ✖ Writes possible alternate forms for algebraic expressions
- ✖ Verbally communicates the use of variables in context
- ✖ Performs mathematical operations accurately - numerical

#### Chapter 6 Graphing relationships and formulae

- This chapter uses patterns to generate formulae, which are then plotted on a graph.
- The terms dependent and independent variables are introduced as well.

#### Chapter 7 Experiencing exponents

- Powers in algebra are introduced using algebraic notation and rules for powers are given.

#### Chapter 14 Solving equations and inequalities

- Solves equations by:
  - ✖ trial and error
  - ✖ balancing
  - ✖ transposing
- Applies the techniques of algebra to solve everyday problems.

### 5. **Mathematics At Work Grade 8 (2000) Human et al. Nasou**

#### Module 1 Mathematic around us

- Introduces the concept of variable and function through flow diagrams and tables.

##### Outcomes

- ✖ Find input and output values
- ✖ find the relationship between the input and output values (find the formula)

#### Module 2 Patterns everywhere.

- The activities mainly consist of filling in tables.

##### Outcomes

- ✖ find the function values

- ✖ Solving equations
- ✖ Describe the behaviour of the function values.
- ✖ Find the rule

#### Module 7 Algebraic language

- All the activities are filling in flow diagrams or tables.

##### Outcomes

- ✖ Translate from flow diagram to an algebraic expression and vice versa
- ✖ Translate from words to an algebraic expression and vice versa
- ✖ Find values of an algebraic expression by substitution.
- ✖ Read values from a table
- ✖ Use brackets and exponential notation.

#### Module 11 Graphs of functions

- Explores functions using graphs. Introduces the co-ordinate system.

##### Outcomes

- ✖ find the function values
- ✖ Solving equations
- ✖ Describe the behaviour of the function values.
- ✖ Find the rule
- ✖ Translate between multiple representations. (words, tables, formula, graph)

#### Module 12 Mathematical communication

- Extensive use is made of tables, adding like terms and removing brackets – only a few examples of each skill.

##### Outcomes

- ✖ Judge whether algebraic expressions are equivalent
- ✖ Form equivalent expressions

#### Module 15 Back to algebra

- Works with functions as models to analyse problems involving two variables. Uses tables once again.

#### Outcomes

- ✗ Appropriate use of notation -  $x$  and  $f(x)$
- ✗ Construct models (formulae like  $y = 2x + 3$ ) for situations
- ✗ Finding output values (function values)
- ✗ Finding input values (solving equations)
- ✗ Analysing the behaviour of function values.

### Module 16 Making life easier

- A few techniques are given for simplifying expressions – removing brackets and adding like terms. The module ends with finding equivalent expressions. This is quite advanced for Grade 8.

#### Outcomes

- ✗ Construct and interpret mathematical models (an algebraic expression or formula or equation.)
- ✗ Construct equivalent forms of the mathematical model. (involves doing simple manipulations)
- ✗ Interpret the different meanings of the equal sign.

### Module 17 Back to equations

- Solving equations using different methods:

Tables method;

Zero difference method;

Inverse operation method

Equivalent equations method.

#### Outcomes

- ✗ Interpret equations without solving them
- ✗ Identify equivalent equations
- ✗ Solve linear equations using different methods



**6. Maths for all Grade 8 (2000) Maths Education Project Macmillan.**

There is an overview of learning algebra on page 55 of teacher's resource book.

**Chapter 1 Mathematical delights.**

- A variety of problems to be solved

**Outcomes**

- ✕ Use your own methods and strategies to solve problems.
- ✕ Draw pictures or make models of the problem situations
- ✕ Organise information on tables to solve problems.
- ✕ Identify and recognise relationships between quantities.
- ✕ Use the relationships between the quantities to solve problems.
- ✕ Use geometric and number patterns to investigate relationships
- ✕ Explain problem situations and solution procedures in your own words.

**Chapter 6 From numbers to letters.**

- A good use of tables with relevant letters and calculations; writing formulae from diagrams.

**Outcomes**

- ✕ Write mathematical sentences using numbers, letters or symbols
- ✕ Read mathematical sentences and explain them
- ✕ Use symbols and letters to stand for any number or unknown quantity
- ✕ Write maths sentences that describe number patterns and relationships
- ✕ Construct mathematical formulae and rules
- ✕ Explain the historical development of a symbolic mathematical language.

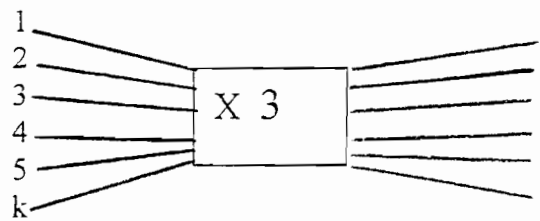
## **APPENDIX 2.**

### **Worksheets and tests.**

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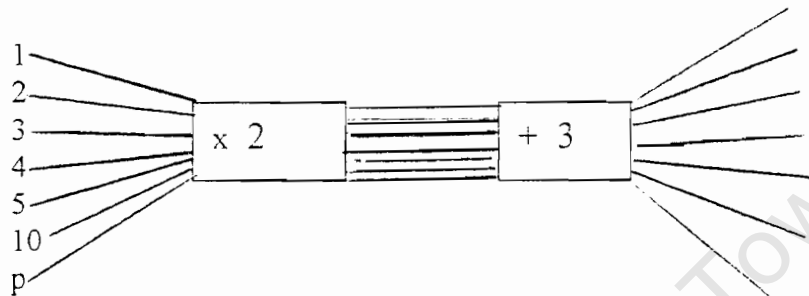
RUSTENBURG GIRLS' HIGH SCHOOL  
GRADE 8 ALGEBRA: Flow diagrams, tables and words.

1. Complete the following flow diagram:



✖ Write the flow diagram in words:

2. Complete the following flow diagram:

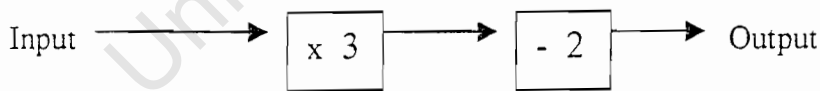


✖ Write the flow diagram in words:

✖ Fill in the information on the table:

Input	1	2	3	4	5	10	p
Output							

3. The flow diagram can also be drawn like this:



✖ Fill in the information on the table:

Input	1	2	3	4	5	10	20	m
Output								

✖ Write the flow diagram in words:

4. Discover the 'rule' of this flow chart from the numbers given in the table and then complete the table:



Input	1	2	3	4	5	10	20		m
Output	5	10	15	20				150	

✖ Write the flow diagram in words:

5. Discover the 'rule' of this flow chart from the numbers given in the table and then complete the table:



Input	1	2	3	4	5	10	20		x	x + 1
Output	5	9	13	17	21	41		101		

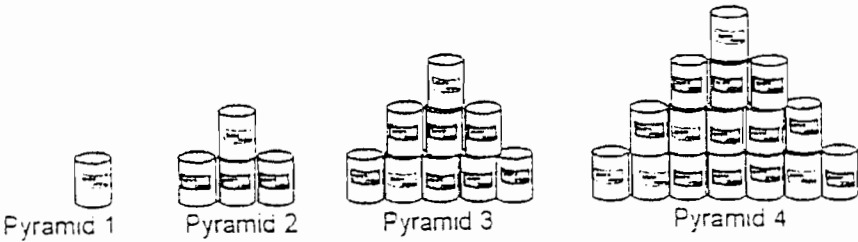
✖ Write the flow diagram in words:

6. In each of the following examples you must complete the table. The diagrams have been given to help you.



Number of coaches	1	2	3	4	5	10	20	n
Number of links	0	1						

✖ Write the relationship between the number of coaches and the number of links in words:



Pyramid Number (p)	1	2	3	4	5	10	20		100	p
Number of cans (c)								625		

✖ Write the relationship between the pyramid number and the number of cans in words:



Number of uprights (t)	1	2	3	4	5	10	20	t
Number of horizontal poles (p)	0	4	8					

✖ Write the relationship between the number of uprights and the number of poles in words:

✖ Number of poles =

✖ p =

- Provide your own examples to illustrate the meaning of the following words:

- |                       |                    |
|-----------------------|--------------------|
| 1 sum                 | 2 difference       |
| 3 multiply            | 4 product          |
| 5 quotient            | 6 twice as many    |
| 7 three times as many | 8 double           |
| 9 half                | 10 triple          |
| 11 consecutive        | 12 more            |
| 13 less               | 14 multiply by     |
| 15 divide             | 16 square a number |
| 17 subtract           | 18 add             |
| 19 greater than       | 20 less than       |
| 21 equal to           |                    |

- Write an (algebraic) expression for each of the following.
- Use variables when necessary.

- |  |                                      |
|--|--------------------------------------|
| 1 the difference between 3 and 1                                     | 4 the sum of 1 and a number          |
| 2 the sum of three consecutive whole numbers starting with 5         | 6 the difference between two numbers |
| 3 the square of 3  | 8 a number divided by 2              |
| 5 the sum of two numbers   | 10 subtract 7 from twice a number    |
| 7 half of 36   |                                      |
| 9 the square of a number   |                                      |
| 11 4 is greater than the sum of two numbers                          |                                      |
| 12 the sum of three times a certain number and double another number |                                      |

- 3 In each case write a verbal expression for the given algebraic expression:

- (a)  $20 \times x + 50$   
 (b)  $10 \times x + 15$   
 (c)  $15 + 10 \times x$   
 (d)  $20 - 3 \times x$   
 (e)  $3 \times x - 20$

Fill in the gaps in these examples, remembering the short-forms for multiplication and division.

1.  $x \text{ --- } \boxed{+3} \text{ --- } ?$

2.  $y \text{ --- } \boxed{\times 2} \text{ --- } ?$

3.  $? \text{ --- } \boxed{-1} \text{ --- } a - 1$

4.  $? \text{ --- } \boxed{\div 4} \text{ --- } \frac{b}{4}$

5.  $p \text{ --- } \boxed{\times 2} \text{ --- } \boxed{+1} \text{ --- } ?$

6.  $? \text{ --- } \boxed{+3} \text{ --- } \boxed{-2} \text{ --- } \frac{k}{3} - 2$

7.  $t \text{ --- } \boxed{+3} \text{ --- } \boxed{\times 4} \text{ --- } ?$

8.  $? \text{ --- } \boxed{+2} \text{ --- } \boxed{\times 3} \text{ --- } 3(p + 2)$

- \* 1. Translate each of the following into an algebraic expression:
- Five added to a certain number.
  - Four less than a number.
  - A number is multiplied by two.
  - Three subtracted from a number.
  - Twelve divided by a number.
  - The sum of eight and a number.
  - The product of a certain number and ten.
  - Half of a number.
  - A number is divided by six.
  - A number is multiplied by two and the product subtracted from eleven.
  - Nine is added to a number and the sum divided by three.
  - Three times a number is divided into eight.
2. Describe each of the following expressions in words:
- |              |                       |                |
|--------------|-----------------------|----------------|
| (a) $p + 3$  | (b) $3 - x$           | (c) $a - 4$    |
| (d) $-3a$    | (e) $\frac{x}{2}$     | (f) $2(y + 1)$ |
| (g) $2y + 1$ | (h) $\frac{a - 1}{5}$ |                |

### Looking different 1

1. Complete the following table

$x$	1	2	3	5	10		
$2x + 5x$							
$3x + 4x$							
$12x - 5$							
$7x$							
$6x + x$							
$9x - 2x$							

- (a) What do you notice in the table?
- (b) Determine the value of  $2x + 5x$  if  $x = 19$ . Discuss your method.
- (c) Determine the value of  $x$  if  $9x - 2x = 35$ . Discuss your method.

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2. Complete the following table.

$x$	1	2	3	4	6	10	
$5x + x$							
$24x - 18x$							
$3x + 2x$							
$6x$							
$4x + 2x$							
$16x - 10x$							

- (a) What do you notice in the table?
- (b) Each of the group members must write a different algebraic expression that will be equivalent to  $24x - 18x$ . Discuss your answers.
- (c) What is the shortest expression that will be equivalent to  $24x - 18x$ ?
- (d) Determine the value of  $16x - 10x$  if  $x = 33,2$ .

We say that expressions in the tables are **equivalent** if they produce the same output numbers for the same input numbers.

Therefore  $4x + 2x$  and  $6x$  are for example **equivalent expressions**, because they produce the same output values for the same input values.

We can explain it in the following way:

$$4x + 2x \text{ means } 4 \times x + 2 \times x = (x + x + x + x) + (x + x) = 6 \times x = 6x$$

3. Choose different algebraic expressions that will be equivalent to  $13x + 6x$ . Complete the table and check if the different expressions are really equivalent.

$x$	1	2	5	12	19	37	45
$13x + 6x$	19	38	95	228	361	703	855

- (a) Determine the value of  $13x + 6x$  if  $x = 13,7$ .
- (b) Determine the value of  $x$  if  $13x + 6x = 703$

4. Complete the table and see if the different expressions are equivalent.

$x$	1	2	5	12	19	37	45
$4x + 13x$							
$13x + 4x$							
$13x + 4$							
$17x$							
$13 + 4x$							
$17 + x$							

- (a) Which of the expressions are equivalent?
- (b) Which of the expressions are not equivalent?

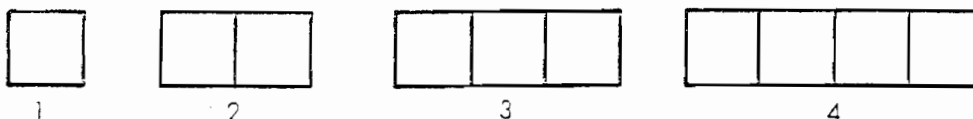
Please Note:

Calculators & Tippex may **NOT** be used.

Question 1

Thandi joins small squares with a side length of 1 cm in order to form a continuous pattern.

The relationship between the perimeter of each shape and the number of squares is shown in the table below:



1.1 With the aid of the above shapes, fill in the values for (a) to (g) (10)

No. of squares		1	2	3	4	.....	(e)	.....	100	<i>m</i>
Perimeter (cm)		(a)	(b)	(c)	(d)	.....	82	.....	(f)	(g)

1.2 Write the above pattern in words. (2)

[12]

Question 2

State whether the following are True or False:

2.1  $a^4 = a + a + a + a$

2.2  $2xy = 2yx$

2.3  $x = 1x^1$

2.4  $3p = p \times p \times p$

2.5  $a \times a \div b = \frac{a^2}{b}$

2.6  $7xyz = 7 \times x \times y \times z$

[1×6=6]



### Question 3

Simplify by selecting an answer from the rectangle below:

3.1  $x + x + x + x + x$

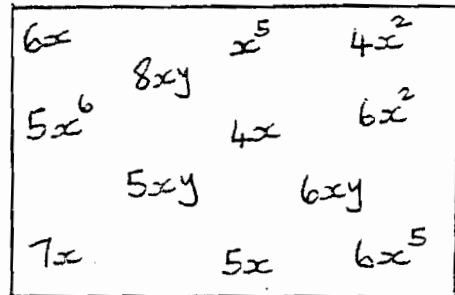
3.2  $6x + x$

3.3  $(3x + x) \times x$

3.4  $3x \times 2y$

3.5  $2x^2 \times 3x^3$

3.6  $2x \times 2y \times 2$



(1)

(1)

(2)

(2)

(2)

(2)

[10]

### Question 4

Write each of the following as an algebraic expression.

- Hint : Let the number be  $p$

4.1 Subtract three from a number

(1)

4.2 Divide a number by three and then add four

(2)

[3]

page 2 of 2

THE END

TOTAL: 31

## ADDING + SUBTRACTING LIKE TERMS

Simplify

- ①  $3x + 4y + 2x + y$
- ②  $4c + 2d + c + 5d$
- ③  $7h + 2 + 3h + 5$
7.  $xy + 4xz + 2xy + 3xz$
9.  $5xy + 3xz + 9xz + 2xy$
11.  $10mn + 11nm + 2m + 3m$
- ⑬  $5x + 3y + 2x + 2y$
15.  $9y + 7z + 3y + 2z$
17.  $7b + 9c + 3b + 2c$
- ⑲  $10xy + 4xz + 9xy + 3xz$
21.  $9ef + 7fg + 2fe + 3gf$
23.  $2y + 3y + 5 + 3y + y + 7 + 2y + 3$
25.  $5a + 3b + 4c + 6a + 3c + 2b + 4a + 4b + 6c$

3. Simplify (if possible)

- |                         |                      |                          |
|-------------------------|----------------------|--------------------------|
| a) $3x^2 + 2x^2$        | b) $c^2 + c^2$       | c) $x^2 + x$             |
| d) $2x^2y + 3x^2y$      | e) $abc + ab$        | f) $6s^2 + 4s^2$         |
| g) $6s^2 + 4s$          | h) $3xy^2 + 2xy^2$   | i) $2xy + 2x$            |
| j) $7a^3 + 3a^3 + 2a^3$ | k) $9x^2 - 5x^2$     | l) $9x^2 - 5x$           |
| m) $10r^3 - 3r^3 - r^3$ | n) $27ab^2 - 19ab^2$ | o) $10a^4 - 5a^4 + 2a^4$ |

## MULTIPLICATION

1. Simplify whole

- |                                   |                             |                                   |
|-----------------------------------|-----------------------------|-----------------------------------|
| a) $x \times y$                   | b) $3x \times 2y$           | c) $x^2 \times y^2$               |
| d) $3x^2 \times 2y^2$             | e) $3x \times 2y \times 4z$ | f) $3x^2 \times 2y^2 \times 4z^2$ |
| g) $5x^3 \times 4y^3$             | h) $5a^2 \times 2b^3$       | i) $3a^3 \times 4b^3$             |
| j) $3a^2 \times 2b^2 \times 4c^2$ | k) $(10ax)(2by)$            | l) $(7ax^2)(2b^2y)$               |

Simplify

- |                                 |                                    |
|---------------------------------|------------------------------------|
| 1. $x^2 \times x^3$             | 31. $10z^5 \times 10z^5$           |
| 4. $2x^3 \times 2x^5$           | 34. $6x^2 \times 2x^4 \times 2x^3$ |
| 7. $3^3 \times 3^{11}$          | 37. $4a^2b \times 2ab^2$           |
| 10. $c^2 \times c^7$            | 40. $5x^3y^2 \times 2x^4y^2$       |
| 13. $4c^2 \times 4c^4$          | 43. $3ab \times 2bc \times 4cd$    |
| 16. $(a)(a^9)$                  | 46. $4x^5 \times 5x^4$             |
| 19. $y^2 \times y^3 \times y^4$ | 49. $7a^{12} \times 12a^7$         |
| 22. $(x^3)(x^5)(x^2)$           | 52. $12a \times a^{12}$            |
| 25. $2p^2 \times 4p^5$          | 55. $(5a^2b)(2a^3b^2)(a^4b^3)$     |
| 28. $7y^3 \times 3y^4$          |                                    |

Simplify whole

- |                                   |  |  |
|-----------------------------------|--|--|
| 1. $a \times a \times a \times a$ | 2. $bbbbbb$                            | 3. $r \times r \times r \times r \times r$ |
| 4. $r + r + r + r + r$            | 5. $3 \times c \times c \times c$      | 6. $3 \times c \times c \times d$          |
| 7. $(5a)(2a)$                     | 8. $(4a)(3a)(2a)$                      | 9. $5 \times 2 \times x \times x$          |
| 10. $pqpqpq$                      | 11. $a \times a \times b \times b$     | 12. $mn \times mn \times mn$               |
| 13. $ab \times ab \times ab$      | 14. $ab + ab + ab$                     | 15. $2ab \times 2ab \times 2ab$            |
| 16. $2ab + 2ab + 2ab$             | 17. $a \times a \times b$              | 18. $3x \times 2y \times 4y$               |
| 19. $a \times a + a \times a$     | 20. $a \times a \times a + a \times a$ | 21. $2xx + 3yy$                            |

## EXTRA EXAMPLES

2.  $5a + 2b + 3b + 4a$
4.  $x + 5 + 2x + 3$
6.  $5 + 2s + 3 + 7s$
8.  $2pq + 3qr + 5pq + 9qr$
10.  $2ab + 3ac + 4ab + 9ac$
12.  $5abc + 3cb + 2bac + bc$
14.  $7a + 4b + 2b + 3a$
16.  $7w + 3x + 4w + 2x$
18.  $6ab + 2ac + 3ab + 2ac$
20.  $5ab + 6ac + 5ab + 6ac$
22.  $2a + 3b + 5c + 5a + 2b + 4c$
24.  $2xy + 3x + 6y + 3xy + 5x$

## MIXED EXAMPLES

1. Simplify (if possible)

- |   |                                  |
|---|----------------------------------|
| a) $2a^2 \times 4a^2$                         | b) $2a^2 + 4a^2$                 |
| c) $10a^3 \times 3a^3$                        | d) $10a^3 + 3a^3$                |
| e) $2a^2b \times 3ab^2$                       | f) $2a^2b + 3ab^2$               |
| g) $5a^2 \times 2b^2 \times 3a^2 \times 4b^2$ | h) $5a^2 + 2b^2 + 3a^2 + 4b^2$   |
| i) $7xy \times 3xy \times 5xy$                | j) $7xy + 3xy + 5xy$             |
| k) $6cd \times 2de \times 3cd \times 4de$     | l) $6cd + 2ce + 3cd + 4ce$       |
| m) $a^2 \times b^2 \times c^2$                | n) $a^2 + b^2 + c^2$             |
| o) $a^2 \times a^3 \times a$                  | p) $a^2 + a^3 + a$               |
| q) $3a^2b \times 4ab^2 \times a^2b$           | r) $3a^2b + 4ab^2 + a^2b$        |
| s) $5x^2 \times 2x \times 3x^2$               | t) $5x^2 + 2x + 3x^2$            |
| u) $6x^2y^2 \times 2x^2y^2 \times 3x^2y^2$    | v) $6x^2y^2 + 2x^2y^2 + 3x^2y^2$ |
| w) $2a^2bc \times 3bac^2 \times 5cba^2$       | x) $2a^2bc + 3bac^2 + 5cba^2$    |

2. Simplify

- |                                     |   |
|-------------------------------------|---|
| a) $2a^2 \times a + 3a^2 \times a$  | b) $4a^3 \times a^4 + 2a^4 \times 3a^3$           |
| c) $5xy \times 2y + 3xy \times 4y$  | d) $7ab \times 2a^2b^2 - 3ab \times 2a^2b^2$      |
| e) $4ab \times 2bc - 2ab \times bc$ | f) $3x^3y^2 \times 2xy^3 + 4xy^3 \times 7x^3y^2$  |
| g) $5a^2 \times 2a \times a - 7a^4$ | h) $7b^2 \times 2b^3 \times b + 4b^2 \times 3b^4$ |
| i) $(2x)(5xy^2) + (4y^2)(6x^2)$     | j) $(3x^2)(2xy^2)(x^3y) + (4x^4)(x^2y)$           |
| k) $(5a^2)(3a^3) + (2a^3)(2a)$      | l) $(6x^2)(4x^3)(0) + (5x^2)(2x^2)$               |

4. Simplify (if possible)

- |                                    |   |
|------------------------------------|---|
| a) $2x^2 + x + 3x^2 + 4x$          | b) $5a^2 + 2a^3 + 3a^2 + 4a^3$          |
| c) $3x^3 + 5x^4 + 4x^3 + 6x^4$     | d) $7y^2 + 4 + 3y^2 + 3$                |
| e) $9x^3 + 5x^2 + 3x^3 + 2x^2$     | f) $6a^3 + 3a^2 + 2a + 4a + 5a^2 + a^3$ |
| g) $6ab^2 + 2a^2b + 4ab^2 + 3a^2b$ | h) $7x^2yz + 2xy^2z + 3xy^2z + x^2zy$   |

2.  $p \times p^3$

5.  $y^2 \times y^{10}$

8.  $3a^3 \times a^{11}$

11.  $4^{20} \times 4^{10}$

14.  $5^5 \times 5^5$

17.  $(d^3)(d^3)$

20.  $x \times x \times x$

23.  $e^4e^2e^3$

26.  $3a^4 \times 3a^5$

29.  $2x \times 3y^8$

32.  $(4a^3)(2a^7)$

35.  $5y^2 \times 3y^3 \times 2y^2$

38.  $6c^2d^2 \times 2c^3d^3$

41.  $2xy^3 \times 10xy$

44.  $3m^4k^4 \times 2k^3m^3$

47.  $6x^5 \times 5x^6$

50.  $3b^9 \times 9b^3$

53.  $6xy \times 2y^3x^2 \times xy$

### DIVISION

Simplify:

8. $\frac{7^4}{7}$	9. $\frac{7a^4}{7a}$	10. $\frac{c^5d^6}{cd^2}$	26. $\frac{x^6y^{12}}{y^4}$
11. $\frac{b^3}{b}$	12. $\frac{a^5}{a^3}$	28. $\frac{a^2b^2}{b^3}$	29. $\frac{4x^3y^3}{x^2y^2}$
14. $\frac{6w^5}{2w^3}$	15. $\frac{6a}{3a}$	31. $\frac{12a^7b^6}{3a^3b^4}$	32. $\frac{6x^3y^5}{3x^2y^2}$
17. $\frac{12p^3}{3p^2}$	18. $\frac{10a^{10}}{5a^6}$	34. $\frac{12a^8b^7c^2}{6a^6b^6c}$	35. $\frac{6a^4b^5c^4}{12a^2bc}$
20. $\frac{ab}{bc}$	21. $\frac{4cde}{2de}$	37. $\frac{15a^2b^2c^2}{12a^2bc^2}$	38. $\frac{13m^4n^4}{13m^3n^2}$
23. $\frac{x^3y^3}{xy}$	24. $\frac{a^4b^5}{a^3b^2}$		

4. $\frac{2^2}{2^3}$	5. $\frac{b^4}{b^2}$
7. $\frac{x^6}{x^7}$	8. $\frac{x^3y^4}{x^2y}$
10. $\frac{a^3b^4c^5}{a^3b^3c^6}$	11. $\frac{x^3y^4z^3}{x^4y^3z^3}$
13. $\frac{12e^8f^6}{6e^4f^7}$	14. $\frac{20a^2b^5}{10a^3b^4}$
16. $\frac{4a^3b}{2a^2c}$	17. $\frac{12de^4}{8d^3e}$
19. $\frac{3}{15b^3}$	20. $\frac{3b^3}{9}$

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### Simplify

1. $\frac{4x+12}{4}$	5. $\frac{8b^3+24b^2}{4b}$
2. $\frac{14b^2+7b+7}{7}$	7. $\frac{a^5+2a^3}{a^2}$
3. $\frac{6a+12b}{6}$	9. $\frac{a^2b+ab^2}{ab}$
4. $\frac{15a^2+5a}{5a}$	11. $\frac{4x^3+8x^3}{2x^2}$
	13. $\frac{12a^2b^3+8a^3b^2+6a^2b^4}{2a^2b^2}$
	15. $\frac{18c^5d^2+9c^3d^3}{9c^3d^4}$

### PRACTICE 13

Simplify

1. $(a^3)^5$	2. $(x^4)^4$	3. $(x^2)^6$
4. $(3a)^2$	5. $(2a)^3$	6. $(5a)^2$
7. $(2a^2)^2$	8. $(3a^3)^2$	9. $(4a^2)^4$
10. $(2ab)^3$	11. $(3ab)^3$	12. $(2x^2y^2)^2$
13. $(3x^3y^3)^2$	14. $(a^3b^6)^4$	15. $(2x^2y^4)^5$

Simplify using the distributive law

1. $2(x+y)$	2. $3(x+2)$	3. $4(a+b)$
4. $3(2a+2b)$	5. $4(3x^2+x)$	6. $10(a+5)$
7. $2(3a+b+2)$	8. $5(2x+2y+2z)$	9. $6(x^3+x^2+x)$
10. $2x(x+y)$	11. $3a(2a+2b)$	12. $4c(2c+3d)$
13. $5b(3b+4)$	14. $3x(5x+2)$	15. $y(3x+3y)$
16. $7c(c+d+1)$	17. $3ab(2a+b)$	18. $10xy(2x+4y)$
19. $5ac(3a+2c)$	20. $12mn(3m+2n)$	21. $3cd(2c+4d+2)$

Simplify fully

1. $2(3a+2b)+3(a+4b)$	2. $5(2x+3y)+6(x+y)$
3. $4c(2c+3)+3c(2c+1)$	4. $5x(4x+y)+2x(3y+2x)$
5. $2x(x+1)+3x^2(4+x)$	6. $7a^2(a^2+a)+3a^2(a^2+a)$
7. $4ab(2a+b)+3ab(3a+2b)$	8. $4a^2b(2a+b)+3ab(4a^2+b)$
9. $3(x^2+2x+1)+4(x^2+5x)$	
10. $7mn(2m^2+4n^2+2)+mn(m^2+n^2)$	

### SUBSTITUTION

6. Determine the values of the following expressions

a) $p^2$ if $p=3$	b) $p^3$ if $p=2$
c) $2p^2$ if $p=4$	d) $p^2+p$ if $p=5$
e) $4p^2q^2$ if $p=2$ and $q=3$	f) $pq^2$ if $p=5$ and $q=2$

7. Determine the values of the following expressions if  $x=2$ ,  $y=3$  and  $z=4$

a) $3z^2$	b) $xyz$	c) $x^2y$
d) $5xz^2$	e) $y^2+z^2$	f) $(y+z)^2$

8. If  $a=2$  and  $b=3$ , what number is represented by

a) $a^3$	b) $9b^2$	c) $(a+b)^2$
d) $a^2b-ab$	e) $(b-a)^3$	f) $4a^3$
g) $(ab)^2$	h) $a^3 \times a^4$	i) $ab^2-b^2a$

### EXTRA EXAMPLES

1. Simplify

a) $4x+2y+2x+y$	b) $15ab+2bc+3ba+cb$
c) $5y+10+3y+2+y$	d) $6a^2+3a+2a^2+4a+5$
e) $5x^2+3x+4+x^2+x+1$	f) $3+5y^3+2y^2+1+3y^2$

2. Simplify

a) $3 \times a \times b$	b) $2 \times a \times a$
c) $5 \times a \times 2 \times c$	d) $p^2 \times p \times p^5$
e) $3a+2a+2b$	f) $3a \times 2a \times 2b$
g) $\frac{x^5}{x^2}$	h) $\frac{x^2}{x^5}$
i) $\frac{4x^4y^5}{2x^2y}$	j) $\frac{8a^3b^4}{2a^4b^2}$
k) $3 \times x^2 \times 2 \times y^2$	l) $a^5 \times b^3 \times a^2 \times b \times 6$
m) $(6x^3y^4)(3x^2y^2)$	n) $(3x^2y^3)^2$

6. $\frac{18x^3+15x^2}{3}$
8. $\frac{9y+12yz}{3y}$
10. $\frac{x^{23}+x^{15}}{x^5}$
12. $\frac{15x^2y^2+5xy^3+20xy^4}{5y^2}$
14. $\frac{14a^4b^3+7a^2b^3}{7a^3b}$

22. $x^2(x+1)$	23. $2y^2(2y+2)$
25. $2x^2(x^2+2x)$	26. $2c^2(3c^3+2c^2)$

GRADE 8  
Time: 50 mins

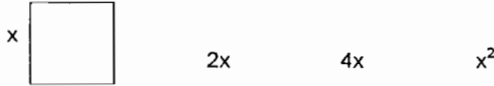
MATHEMATICS TEST  
ALGEBRA practice test

APRIL 2002  
Marks: 55

QUESTION 1

Choose the correct answer from those given. Just write the answer on your page.

- 1.1 The expression: add two to a number and then multiply by six can be written as  $12 + n$   $6(2 + n)$   $2 + 6n$
- 1.2  $6a^3 - 2a^3 =$  4  $4a^3$   $4a$
- 1.3  $2aaabbbb =$   $2a^3b^4$   $2ab$   $14ab$
- 1.4 If  $a = 3$  and  $b = 2$  then  $2ab^2 =$  72 24 144
- 1.5 The perimeter of the square when each side is  $x$  is



[5]

QUESTION 2

Decide whether the following are true or false. Just write down true or false:

- 2.1  $e \times e \times e \times e \times e = 5e$  2.2  $a + a + a + a + a = 5a$
- 2.3  $\frac{2p^3}{4p} = \frac{p^2}{2}$  2.4  $x + x \times x + x = 4x^2$
- 2.5  $3(x - 1)$  is one term 2.6  $2a^2b$  and  $3ba^2$  are like terms
- 2.7  $3x^2 - 2x + 1$  is an expression written in descending order
- 2.8  $3x^2 - 2x + 1$  is called a binomial
- 2.9 In the expression  $3a \times 4 + 2 \times 5a$   $4 + 2$  must be calculated first.
- 2.10  $\frac{3x + 2x}{5} = x$

[10]

QUESTION 3

Simplify the following if possible:

- 3.1  $3p + 2q + 4p - q + 3q$  3.2  $a^2 + a^2$
- 3.3  $3p^2 \times 6p^3$  3.4  $2ab \times 3ab^2 \times a^2b$
- 3.5  $3x \times 2y + 5x \times y$  3.6  $\frac{3a^2b}{abc}$
- 3.7  $\frac{2x^2y^3z}{6xy^5z}$  3.8  $\frac{5ab + 10a^2b^3}{5ab}$
- 3.9  $(3ab^3)^2$  3.10  $3(2x + 4y)$

[25]

QUESTION 4

Consider the expression:  $x^2 - 2x^3 - 7x + 15$

- 4.6 How many terms are there in the expression?
- 4.7 What is the coefficient of the first term?
- 4.8 Write down the term with the highest power.
- 4.9 Find the value of the expression if  $x = 1$ .
- 4.10 Rewrite the expression in descending order.

[7]

QUESTION 5

Copy this table and complete it by putting the values for  $x$  in the expressions:

$x$	1	2
$5x^2$		
$(3x + 2)$		

[4]

QUESTION 6

Take a number, call it  $x$   
Multiply it by 2  
Add 2 to the answer  
Multiply that number by 3  
Subtract 3 from your answer  
Divide that answer by 3  
Subtract 1  
Divide by 2.  
Your answer should be  $x$ !

**QUESTION 1**  
Choose the correct answer from those given. Just write the answer on your page.

- 1.1  $a + a + a =$       3a       $a^3$        $3a^3$       a
- 1.2  $2a + 3b =$       5ab      6ab       $2a + 3b$        $5a^2b^3$
- 1.3  $(xy)^2 =$       2xy       $xy^2$        $x^2y^2$        $2x^2y^2$
- 1.4  $2mn + 5nm =$       7mn      10mn       $7m^2n^2$        $2mn + 5nm$
- 1.5  $2(3x + 4y) =$       14xy       $6x + 4y$        $49x^2y^2$        $6x + 8y$
- [5]

**QUESTION 2**  
Decide whether the following are true or false. Just write down true or false:

- 2.1  $2^3 \times 2^4 = 2^7$       2.2  $p \times p \times p \times p \times p = p^5$
- 2.3  $\frac{x^3}{x} = 1^3$       2.4  $2a^2 = 36$  if  $a = 3$
- 2.5  $3b^2 + 2b^2 = 5b^4$
- [5]

**QUESTION 3**  
Write in algebra:

3.1 add six to a number

3.2 subtract four from a number

3.3 add one to a number and then divide by three.

[4]

**QUESTION 4**  
Consider the expression:  $5x^3 - 2x + 4x^2 + 3$

4.1 How many terms are there in the expression?

4.2 What is the coefficient of the second term?

4.3 What is the constant term?

4.4 Find the value of the expression if  $x = 2$ .

4.5 Rewrite the expression in ascending order.

[8]

**QUESTION 5**  
Simplify the following if possible:

- 5.1  $5x + 2y + 3x - y$       5.2  $3x^2 \times 4x^3$
- 5.3  $(2x^3y^2)^2$       5.4  $(4a^7)(2a^3)$
- 5.5  $2p(3p^2 + 5q)$       5.6  $2(x+1) + 4(2x+3)$
- 5.7  $\frac{12x^3}{2x^2}$       5.8  $\frac{3a^3bc^4}{6a^2b^2c}$
- 5.9  $\frac{6x + 4x^2}{2x}$       5.10  $2a^3 \times a^2 + a \times 4a^4$
- 5.11  $(3a^2 + 2a^2) + 5a + (4a - 3a) \times 2$

[34]

**QUESTION 6**  
Copy this table and complete it by putting the values for x in the expressions:

x	0	4
$2x^2 + 1$		
$\frac{x + 4}{2}$		

[4]

Question 1:

Determine the value of the following WITHOUT using a calculator:

1.1  $(9) + (-14) - (-2)$  (2)

1.2  $-10 - (7)(3)$  (2)

[4]

Question 2:

2.1 Find any fraction that lies between  $\frac{2}{3}$  and  $\frac{3}{4}$ . (3)

2.2 Which fraction is bigger,  $\frac{2}{5}$  or  $\frac{3}{8}$ ? (2)

[5]

Question 3:

Simplify, WITHOUT using a calculator:

3.1  $\frac{3}{2p} + \frac{7}{5p}$  (2)

3.2  $\frac{4a}{12c} \times \frac{24c}{8a}$  (2)

3.3  $-\left(\frac{2x}{5}\right)^2 \div \left(\frac{4}{3}\right)^2$  (4)

3.4  $\left(2\frac{1}{4} + 1\frac{1}{4}\right) - \left(2\frac{1}{4} - 1\frac{1}{2}\right)$  (3)

3.5  $\frac{\frac{3x}{4} + \frac{2x}{3}}{\frac{x}{2} - \frac{x}{3}}$  (4)

[15]

Question 4:

Simplify where necessary, WITHOUT using a calculator and write the answer in scientific notation:

4.1 536 (1)

4.2 83 0000 (1)

[2]

Question 5:

Write the following number in full and then express it in words:

$6,85 \times 10^8$  (2)

[2]

Question 6:

Simplify:

6.1  $x.x.x.x$  (1)

6.2  $y^2 + y^2$  (1)

6.3  $-2x^2 + 7x^3 + 6x^2 - 2x^3 - x^3$  (2)

6.4  $(3y^5)^2$  (2)

6.5  $4x(2x)^3$  (2)

6.6  $-4a^2bc^2 \times -3a^3bc^2$  (4)

6.7  $-2(x+2) + (x-4) + 3$  (3)

6.8  $-4x^2(2x^2y - 3xy^3)$  (3)

6.9  $\frac{2a+3a}{5a} - (2a-a)$  (3)

[21]

Question 7:

7.1 Subtract  $-3a^2 + 8ab - 4b^2$  from  $9a^2 - 8ab + 7b^2$  (4)

7.2 Divide  $9x^3y^4 + 6x^5y^4 - 3x^2y^3$  by  $-3x^2y^3$  (4)

[8]

Question 8:

Consider the following expression :

$$8x + 7x^3 - 2x^2 + 4$$

- 8.1 Write down the term with the largest numerical coefficient. (1)  
8.2 Write down the term with the highest power of  $x$ . (1)  
8.3 What is the coefficient of  $x^2$  ? (1)  
8.4 What is the value of the third term if  $x = 2$  ? (2)  
8.5 Rearrange the expression in descending powers of  $x$ . (2)  
[7]

Question 9:

If  $a = 2$ ,  $b = 3$  and  $c = -1$ , calculate the value of :

- 9.1  $2(b - c) + a$  (3)  
9.2  $\frac{ab^2}{2c}$  (2)  
[5]

Question 10:

Solve for  $x$  :

- 10.1  $-x - \frac{3}{5} = 1$  (2)  
10.2  $\frac{5x}{-2} = 25$  (2)  
10.3  $5x - 7 + 2x = 3x + 11 + 6$  (3)  
10.4  $5(x - 1) - (2x + 1) = 2(x - 2)$  (5)

[12]



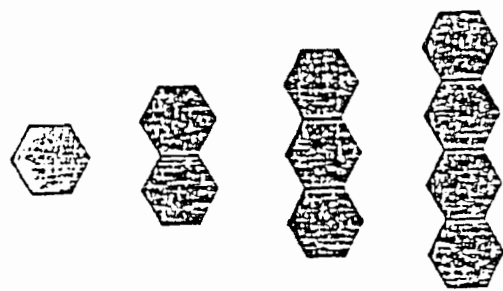
Question 11:

Rewrite each of the following into an algebraic expression:

- 11.1 the sum of  $a$  and  $b$  equals 8 (1)
- 11.2 four times  $x$  equals twice  $y$  (1)
- 11.3 subtract  $c$  from the product of  $a$  and  $b$  (2)
- 11.4 the square of the difference between  $a$  and  $b$  equals 4 (2)
- [6]

Question 12:

In a certain restaurant hexagonal tables are used. One person can be seated on each side of a table. To accommodate larger groups, tables are linked as shown in the diagram below:



Complete the table below with the aid of the diagram:

No. of Tables	1	2	3	4	10	$n$
No. of people	6	(a)	(b)	18	(c)	(d)

(6)

[6]

Question 13:

Simplify each of the following ratios below:

13.1     25 ml : 1 l (1)

13.2     R2,50 : R10,00 (1)

13.3     100 min :  $1\frac{3}{4}$  hr (2)

[4]

Question 14:

(Where necessary round off answers to two decimal places)

14.1     Decrease 32 in the ratio 3:4 (2)

14.2     Two girls share 52 sweets in the ratio 4:9. How many sweets does each receive ? (3)

14.3     Alice feeds her dog of mass 15kg a total of 4 cups of food a day. Calculate how many cups Joanna's rotweiler must be fed if he has a mass of 25kg. (3)

14.4     A clothing store advertised 20% less on all jeans. What will the new price of a jeans which was R89 be ? (2)

14.5     A point P divides a line segment AB in the ratio 4:3. If AB is 35 cm, what is the length of BP ? (3)

14.6     If a car travels at an average speed of 90 km/hr it takes  $4\frac{1}{2}$  hrs to complete a journey. If the average speed was increased to 100 km/hr, how long would the same journey take ? (4)

14.7 Lauren needs to make brownies for her 35 classmates. However, her recipe will only make 20.

14.7.1 What is the ratio by which she needs to adjust her recipe ? (1)

14.7.2 Listed below are all the ingredients for 20 brownies. Calculate the amount of milk and cocoa powder she will need in order to make 35 brownies:

250 g cake flour

65 ml milk

40g butter

20 g cocoa powder

56 g coconut

(2)

14.7.3 By mistake Lauren uses 40g of cocoa powder instead of the amount needed to make 35 brownies. By what ratio should she now adjust all the other amounts in order to rectify the mistake ?  
( \* she will now make more brownies )

(3)

[23]

TOTAL :120

THE END